REFLECTION ARRANGEMENTS AND RIBBON REPRESENTATIONS

ALEXANDER R. MILLER

ABSTRACT. Ehrenborg and Jung [15] recently related the order complex for the lattice of d-divisible partitions with the simplicial complex of *pointed ordered set partitions* via a homotopy equivalence. The latter has top homology naturally identified as a Specht module. Their work unifies that of Calderbank, Hanlon, Robinson [13], and Wachs [34]. By focusing on the underlying geometry, we strengthen and extend these results from type A to all real reflection groups and the complex reflection groups known as *Shephard groups*.

1. INTRODUCTION

The aim of this paper is to elucidate a phenomenon that has been studied for the symmetric group $\mathfrak{S}_n$ by studying the underlying geometry. Here we sketch the phenomenon, along with our geometric interpretation and generalization.

For $n+1$ divisible by d, recall that the d-divisible partition lattice $\Pi^d_{n+1} \cup \{\hat{0}\}$ is the poset of partitions of the set $\{1, 2, \ldots, n+1\}$ with parts divisible by d, together with a unique minimal element $\hat{0}$ when $d > 1$. In [13], Calderbank, Hanlon and Robinson showed that for $d > 1$ the top homology of the order complex $\Delta(\Pi^d_{n+1} \setminus \{\hat{1}\})$, when restricted from $\mathfrak{S}_{n+1}$ to $\mathfrak{S}_n$, carries the *ribbon representation* of $\mathfrak{S}_n$ corresponding to a ribbon with row sizes $(d, d, \ldots, d, d-1)$. Wachs [34] gave a more explicit proof of this fact. Their results generalized Stanley's [29] result for the Möbius function of $\Pi^d_n \cup \{\hat{0}\}$, which in turn generalized G. S. Sylvester's [31] result for 2-divisible partitions $\Pi^2_n \cup \{\hat{0}\}$.

Ehrenborg and Jung extend the above results by introducing posets of *pointed partitions* $\Pi^\bullet_{\vec{c}}$ parametrized by a composition $\vec{c}$ of n with last part possibly 0, from which they obtain all ribbon representations. More importantly, they explain why Specht modules are appearing by establishing a homotopy equivalence with another complex whose top homology is naturally a Specht module.

Ehrenborg and Jung construct their pointed partitions $\Pi^\bullet_{\vec{c}} \subset \Pi^\bullet_n \cong \Pi_{n+1}$ by distinguishing a particular block (called the *pointed block*) and restricting to those of type $\vec{c}$. They show that $\Delta(\Pi^\bullet_{\vec{c}} \setminus \{\hat{1}\})$ is homotopy equivalent to a wedge of spheres, and that the top reduced homology $\widetilde{H}_{\text{top}}(\Delta(\Pi^\bullet_{\vec{c}} \setminus \{\hat{1}\}))$ is the $\mathfrak{S}_n$-Specht module corresponding to $\vec{c}$.

Their approach is to first relate $\Pi^\bullet_{\vec{c}}$ to a selected subcomplex $\Delta_{\vec{c}}$ of the simplicial complex $\Delta^\bullet_n$ of ordered set partitions of $\{1, 2, \ldots, n\}$ with last block possibly empty. In particular, they use Quillen's fiber lemma to show that $\Delta(\Pi^\bullet_{\vec{c}} \setminus \{\hat{1}\})$ is homotopy equivalent to $\Delta_{\vec{c}}$. They then give an explicit basis for $\widetilde{H}_{\text{top}}(\Delta_{\vec{c}})$ that identifies the top homology as a Specht module.

Ehrenborg and Jung recover the results of Calderbank, Hanlon and Robinson [13] and Wachs [34] by specializing to $\vec{c} = (d, \ldots, d, d-1)$.

Taking a geometric viewpoint, one can consider $\Delta_n^\bullet$ as the barycentric subdivision of a distinguished facet of the standard simplex having vertices labeled with $\{1, 2, \ldots, n, n+1\}$. As such, it carries an action of $\mathfrak{S}_n$ and is a balanced simplicial complex, with each $\Delta_{\vec{c}}$ corresponding to a particular type-selected subcomplex. Under this identification, the poset $\Pi_{\vec{c}}^\bullet$ corresponds to linear subspaces spanned by faces in $\Delta_{\vec{c}}$.

We propose an analogous program for all well-generated complex reflection groups by introducing *well-framed* and *locally conical* systems. We complete the program for all irreducible *finite* groups having a presentation of the form

$$(1) \qquad \langle r_1, \ldots, r_\ell \mid r_i^{p_i} = 1, \quad \underbrace{r_i r_j r_i \ldots}_{q_{ij}} = \underbrace{r_j r_i r_j \ldots}_{q_{ij}} \quad i < j \rangle$$

with $p_i \geq 2$ for all i and subject to the constraint that $p_i = p_j$ whenever q_{ij} is odd. Each such group has an irreducible faithful representation as a complex reflection group. The irreducible finite Coxeter groups are precisely those with each $p_i = 2$, i.e., those with a real form. The remaining groups are *Shephard groups*, the symmetry groups of regular complex polytopes.[1] The family of Coxeter and Shephard groups contains 21 of the 26 exceptional well-generated complex reflection groups. Using Shephard and Todd's numbering, the remaining five groups are $G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$.

Outline. Solomon's ribbon/descent representations and their analogues appear naturally as homology representations within certain subcomplexes of a Coxeter-like complex $\Delta(W, R)$, and the idea of this paper is to transfer the ribbon/descent representations from $\Delta(W, R)$ to the lattice of intersections of reflecting hyperplanes for W using the map that takes each face to its linear span. The difficulty is in showing that the appropriate restrictions of this map do actually transfer the corresponding homology representations, in the strong sense that they define equivariant homotopy equivalences.

The first two sections set up a general framework for well-generated reflection groups. Roughly speaking, Section 2 focuses on $\Delta(W, R)$, and Section 3 focuses on the transfer. §2.1 gathers some preliminaries. §2.2 introduces *well-framed* and *strongly stratified* systems (W, R, Λ), then §2.3–2.5 develop and connect the algebra and geometry of these systems. The vectors of the *frame* Λ play the role of fundamental weights for a Weyl group; the W-translates of the real hull of these vectors embed nicely inside unitary space to form a geometric realization of the abstract Coxeter-like complex $\Delta(W, R)$, whose faces are indexed by standard parabolic cosets.

§3.1 introduces the main objects of the paper: certain subcomplexes Δ_T^U of $\Delta(W, R)$ and certain subposets Π_T^U of the lattice of reflecting hyperplane intersections for W, where U and T are subsets of the set of generators R. The Δ_T^U's will naturally carry the ribbon/descent representations when $\Delta(W, R)$ is Cohen–Macaulay, and the Π_T^U's are the images of the Δ_T^U's under the map that takes a face to its linear span. Ehrenborg and Jung's pointed objects correspond to the special case when $W = \mathfrak{S}_n$ and $U = \{(n-1, n)\}$, and their result says that, in this case one has that for all T the top homology groups of Δ_T^U and $\Pi_T^U \setminus \{\hat{1}\}$ are isomorphic as $\mathfrak{S}_{n-1}$-modules; we elaborate on this specialization in §3.4, where we give a detailed dictionary between our geometric language and their combinatorial language. In §3.2 and §3.3 we prepare a much stronger and more general version of the type A_n phenomenon

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[1] The algebraic unification of Coxeter groups and Shephard groups presented here does not appear to be widely known, and is attributed to Koster [20, p. 206].

for use within our general framework of well-framed systems from Section 2. §3.2 isolates well-framed systems with a special property which guarantees that not only are the top homology modules of Δ_T^U and $\Pi_T^U \setminus \{\hat{1}\}$ isomorphic for all T and nonempty U, but in fact Δ_T^U and $\Pi_T^U \setminus \{\hat{1}\}$ are equivariantly homotopy equivalent. This is Theorem 3.7, and the special systems involved are called *locally conical* systems. §3.3 applies homology to the homotopy equivalence (Theorem 3.7) in order to transfer the homology representations of the Δ_T^U's onto the $\Pi_T^U \setminus \{\hat{1}\}$'s (Theorem 3.8), then focuses on the case when $\Delta(W, R)$ is Cohen–Macaulay, so that the homology of Δ_T^U is concentrated in the top dimension and its character can be written as an alternating sum of principal characters induced from standard parabolics by applying the Hopf-trace formula. The precise description is Theorem 3.11, and the homology representations are *ribbon representations* of $W_{R \setminus U}$.

In Section 4 we show that all of our results from Section 3 apply to finite irreducible Coxeter groups. In particular, the main theorem (Theorem 4.1) of this section tells us that Theorem 3.7, Theorem 3.8, and Theorem 3.11 apply to finite irreducible Coxeter systems (W, R) when Λ is chosen from the extreme rays of a Weyl chamber, so that $\Delta(W, R)$ is the Coxeter complex. The object here is to prove that (W, R) is locally conical, for which we introduce an iterative method of detecting cone points based on chamber geometry.

Section 5 gives the Shephard analogue of Theorem 4.1. The analogue is Theorem 5.3, and the upshot is that all previous results of Section 3 apply not only to finite irreducible Coxeter groups, but to Shephard groups as well. In particular, Theorem 3.7, Theorem 3.8, and Theorem 3.11 apply to *Shephard systems* (W, R). The main difficulty again lies in proving that the systems are locally conical. This requires a different method of proof than in the Coxeter case.

In Section 6 we study some other aspects of ribbon representations for Shephard groups, which had previously been studied in the Coxeter case by Solomon, Stembridge, and Reiner. The main results are Theorem 6.1 , Theorem 6.2, and Theorem 6.3. The first two generalize theorems of Solomon for Coxeter groups to the entire (finite irreducible) Coxeter–Shephard family, and our proof of the second theorem is in fact much simpler than Solomon's original proof. Theorem 6.3 gives a determinantal expression for a multivariate graded inner product that describes the decomposition of the coinvariant algebra into ribbon representations, which had been studied in the Coxeter case by Reiner following Stembridge.

Section 7 concludes the paper with some miscellaneous results, questions, and conjectures.

2. Frames

In this section we introduce special sets of vectors for well-generated reflection groups that play the role of fundamental weights for Weyl groups.

2.1. **Preliminaries.** Let V denote an ℓ-dimensional $\mathbb{C}$-vector space. A *reflection* in V is any non-identity element $g \in \mathrm{GL}(V)$ of finite order that fixes some hyperplane H, and a finite group $W \subset \mathrm{GL}(V)$ is called a *reflection group* if it is generated by reflections. Henceforth, we assume that W acts irreducibly on V. Shephard and Todd gave a complete classification of all such groups in [25]; see [18] for the general theory of complex reflection groups.

Given a (finite) reflection group $W \subset \mathrm{GL}(V)$, we may choose a positive definite Hermitian form $\langle -, - \rangle$ on V that is preserved by W, i.e.,

$$\langle gx, gy \rangle = \langle x, y \rangle$$

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for all $x, y \in V$ and $g \in W$. We always regard V as being endowed with such a form, which is unique up to positive real scalar when W acts irreducibly on V. We let $|-|$ denote the associated norm, so that $|v|^2 = \langle v, v \rangle$ for all $v \in V$.

As a special case of a complex reflection group, consider a finite group $W \subset \mathrm{GL}(\mathbb{R}^\ell)$ that is generated by reflections through hyperplanes. By extending scalars, we consider W as acting on $\mathbb{C}^\ell$, and regard W as a reflection group. We call a (complex) reflection group that arises in this way a *(finite) real reflection group*.

A subgroup $W \subset \mathrm{GL}(V)$ naturally acts on the dual space V^* via $gf(v) = f(g^{-1}v)$, and this action extends to the symmetric algebra $S = S(V^*)$. An important subalgebra of S is *the ring of invariants* S^W, whose structure in fact characterizes reflection groups:

Theorem 2.1 (Shephard–Todd, Chevalley)**.** *A finite group $W \subset \mathrm{GL}(V)$ is a reflection group if and only if S^W is a polynomial algebra $\mathbb{C}[f_1, f_2, \ldots, f_\ell]$.*

When W is a reflection group, such a set $f_1, f_2, \ldots, f_\ell$ of algebraically independent homogeneous polynomials generating S^W is not unique, but we do have uniqueness for the corresponding *degrees* $d_1 \leq d_2 \leq \ldots \leq d_\ell$.

It is well known that the minimum number of reflections required to generate W is either $\dim V$ or $\dim V + 1$. If there exists such a generating set R with $|R| = \dim V$, then we say that W is *well-generated* and that (W, R) is a *well-generated system*. (Finite) real reflection groups form an important class of well-generated reflection groups. Another important family consists of symmetry groups of *regular complex polytopes*; these are known as *Shephard groups*, and were extensively studied by both Shephard [24] and Coxeter [14].

The (complex) reflecting hyperplanes for a real reflection group W cut out spherical simplices on the unit sphere

$$\mathbb{S}^{\ell-1} = \{x \in \mathbb{R}^\ell \subset \mathbb{C}^\ell \ : \ |x| = 1\}$$

to form the *Coxeter complex*, a simplicial complex with many wonderful properties. We call the maximal simplices of a Coxeter complex *chambers*. The real cone $\mathbb{R}_{\geq 0} C$ over a chamber C is called a *(closed) Weyl chamber*, and one nice feature of the Coxeter complex is its algebraic description when (W, R) is a *simple system*, i.e., when R is the set of reflections through walls of a Weyl chamber. In this case, the poset of faces for the complex has the alternate description [1, §1.5.9] as the poset of standard parabolic cosets

$$\Delta(W, R) = \{gW_J \ : \ g \in W, J \subseteq R\},$$

ordered by reverse inclusion, i.e.,

$$gW_J < g'W_{J'} \quad \text{if} \quad gW_J \supset g'W_{J'}$$

where $W_J = \langle J \rangle$.

Naturally, one can define such a poset $\Delta(W, R)$ for an arbitrary group W with distinguished set of generators R. In [3], Babson and Reiner show that the geometry of this general construction is still well behaved when R is finite and minimal with respect to inclusion. In particular, they show that such posets $\Delta(W, R)$ are *simplicial posets*, meaning that every lower interval is isomorphic to a Boolean algebra. In fact, each $\Delta(W, R)$ is *pure* of dimension $|R| - 1$ and *balanced*. In other words, there is a coloring of the atoms using $|R|$ colors so that each maximal element lies above exactly one atom of each color. The natural coloring given by

$$gW_{R \setminus \{r_i\}} \longmapsto \{r_i\}$$

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extends to a *type function*

$$\text{type} : \Delta(W, R) \longrightarrow \{\text{subsets of } R\}$$
$$gW_{R \setminus J} \longmapsto J$$

and so for each subset $T \subseteq R$ we have the subposet *selected by T*:

$$\Delta_T(W, R) \overset{\textbf{def}}{=} \{\sigma \in \Delta(W, R) \ : \ \text{type}(\sigma) \subseteq T\}$$
$$= \{gW_{R \setminus J} \ : \ J \subseteq T\}.$$

We will often write Δ in place of $\Delta(W, R)$, and Δ_T in place of $\Delta_T(W, R)$.

When W is a reflection group, the lattice of intersections of reflecting hyperplanes for W under reverse inclusion is denoted by $\mathscr{L}_W$, or simply $\mathscr{L}$. It is a subposet of the lattice $\mathscr{L}(V)$ of all $\mathbb{C}$-linear subspaces of V ordered by reverse inclusion. For a subset $A \subseteq V$, let

$$\text{Span}(A) \overset{\textbf{def}}{=} \text{minimal } \mathbb{C}\text{-linear subspace of } V \text{ containing } A.$$

$$\text{AffSpan}(A) \overset{\textbf{def}}{=} \text{minimal } \mathbb{C}\text{-linear affine space of } V \text{ containing } A.$$

$$\text{Hull}(A) \overset{\textbf{def}}{=} \left\{ \sum_{i=1}^{m} t_i a_i \ : \ a_i \in A, \ t_i \geq 0, \ \sum_{i=1}^{m} t_i = 1 \right\}.$$

These are the only notions of span and hull that will appear in this paper.

2.2. Frames and well-framed systems.

Definition 2.2. For (W, R) a well-generated system, a *frame*

$$\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_\ell\}$$

is a collection of nonzero vectors such that

$$\lambda_i \in H_1 \cap \ldots \cap \hat{H}_i \cap \ldots \cap H_\ell \quad \text{for} \quad 1 \leq i \leq \ell,$$

where H_i is the reflecting hyperplane of $r_i \in R$. We say that (W, R, Λ) is a *framed system*.

We will sometimes index a generating set R and frame Λ with $\{0, 1, \ldots, \ell - 1\}$ instead of $\{1, 2 \ldots, \ell\}$, so that $R = \{r_0, r_1, \ldots, r_{\ell-1}\}$ and $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_{\ell-1}\}$, when W is the symmetry group of a regular polytope and Λ is chosen from the vertices of the barycentric subdivision in such a way that each λ_i corresponds to an i-dimensional face of the polytope; see Figures 4, 5, and Section 5.1.

Before stating the next definition, we make precise what is meant by (piecewise) $\mathbb{R}$-linearly extending a map $F : \text{Vert}(\Delta) \to V$ on vertices of a simplicial poset Δ. Identify Δ with a CW-complex as in [8, §12], so that each point $y \in \Delta$ is contained in a unique (open) cell σ. Since Δ is a simplicial poset, the cell admits a characteristic map f that maps the standard $(\dim \sigma)$-simplex onto $\overline{\sigma}$ while restricting to a bijection between vertices (0-cells), and hence there are unique scalars c_v such that

$$f^{-1}(y) = \sum_{v \in \text{Vert}(\overline{\sigma})} c_v f^{-1}(v), \qquad \sum_{v \in \text{Vert}(\overline{\sigma})} c_v = 1, \quad \text{and} \quad c_v \in \mathbb{R}_{\geq 0} \text{ for all } v.$$

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We extend F to all of Δ by defining

$$F(y) = \sum_{v \in \text{Vert}(\overline{\sigma})} c_v F(v).$$

The following definition is central to what follows.

Definition 2.3. Let (W, R, Λ) be a framed system. Define $\rho : \Delta(W, R) \longrightarrow V$ by the following map on vertices

$$gW_{R \setminus \{r_i\}} \xmapsto{\rho} g\lambda_i$$

(which is well defined because the subgroup $W_{R \setminus \{r_i\}}$ fixes λ_i) and then extending $\mathbb{R}$-linearly over each face $gW_{R \setminus J}$ of Δ, so that

$$gW_{R \setminus J} \xmapsto{\rho} g\Lambda_J$$

for

$$\Lambda_J := \text{Hull}(\{\lambda_i \ : \ r_i \in J\}).$$

- We say that (W, R, Λ) is *well-framed* if the equivariant map ρ is an embedding.
- If, in addition, each $X \in \mathscr{L}_W$ contains the image of at least one $(\dim X - 1)$-face under ρ, then we say that (W, R, Λ) is *strongly stratified*.

Convention 2.4. Given a well-framed system (W, R, Λ), we will often identify the abstract simplicial poset $\Delta(W, R)$ and the geometric realization $\rho(\Delta(W, R))$ afforded by ρ.

Observe that a system (W, R, Λ) is well-framed if and only if for any positive real constants $c_1, c_2, \ldots, c_\ell$, the system $(W, R, \{c_1\lambda_1, c_2\lambda_2, \ldots, c_\ell\lambda_\ell\})$ is well-framed.

Example 2.5. Let (W, R) be a simple system, i.e., an irreducible finite real reflection group W with R the set of reflections through walls of a Weyl chamber. If W is a Weyl group, one can obtain a real frame $\Lambda \subset \mathbb{R}^\ell$ by choosing Λ to be the set of fundamental dominant weights. More generally, a real frame is obtained by choosing one nonzero point on each extreme ray of a Weyl chamber corresponding to R. The resulting system (W, R, Λ) is strongly stratified and $\rho(\Delta(W, R))$ is homeomorphic to the Coxeter complex via radial projection; Figure 1 illustrates the construction for $W = I_2(5)$, the dihedral group of order 10.
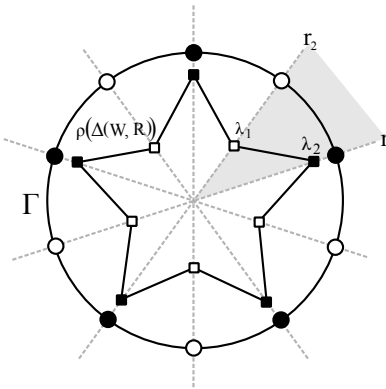

FIGURE 1. A well-framed system and shaded Weyl chamber for $I_2(5)$.

Definition 2.6. We say that (W, R) is *well-framed* or *strongly stratified* if there exists a frame Λ for which (W, R, Λ) is well-framed or strongly stratified, respectively.

For W a real reflection group, the following example suggests that only simple systems (W, R) can be well-framed by a real frame $\Lambda \subset \mathbb{R}^\ell$. In other words, the well-framed systems (W, R, Λ) with Λ real and W a (complexified) finite real reflection group are completely characterized in Example 2.5; see Proposition 2.10 below.

Example 2.7. Let $W = I_2(5)$, the dihedral group of order 10, and let $R = \{r_1, r_2\}$, where r_1, r_2 are the reflections indicated in Figure 2. In this case, (W, R) is not a simple system because the corresponding hyperplanes H_1, H_2 do not form a Weyl chamber; see the shaded region. Two real frames are shown in the figure, one with $|\lambda_1| \neq |\lambda_2|$ (left) and one with $|\lambda_1| = |\lambda_2|$ (right). Both fail to give a well-framed system. For example, when $|\lambda_1| = |\lambda_2|$ (on the right in Figure 2) the map $\|\Delta(W, R)\| \xrightarrow{\rho} \rho(\Delta)$ is a double covering.
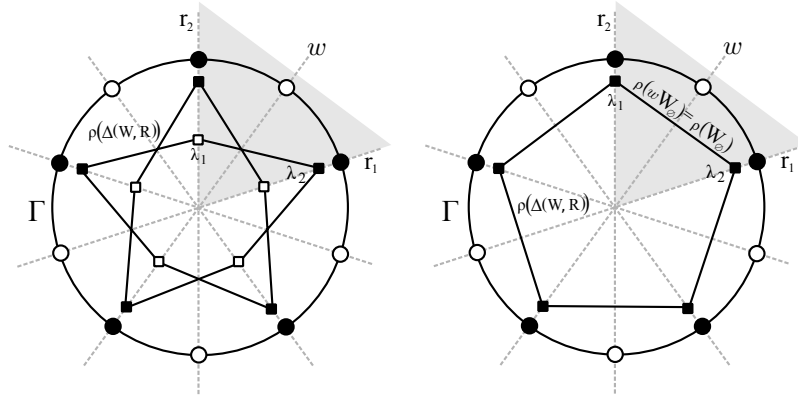

FIGURE 2. Systems (W, R, Λ) that are not well-framed for $W = I_2(5)$, $R = \{r_1, r_2\}$, and real $\Lambda = \{\lambda_1, \lambda_2\}$.

Figures 1 and 2 show that for a fixed W, it is possible to have both good and bad choices for R and Λ. They also show that (W, R, Λ) being well-framed is not a local property, by which we mean that for $\ell(w)$ large, one has to check for intersections of the simplices $\rho(eW_\varnothing)$ and $\rho(wW_\varnothing)$; see Figure 2.

Although some systems (W, R) do not give a well-framed triple (W, R, Λ) for any *real* $\Lambda \subset \mathbb{R}^\ell$, one may still be able to choose a good frame $\Lambda \subset \mathbb{C}^\ell$, as the next example shows.

Example 2.8. Let $W = I_2(5)$ and let $R = \{r_1, r_2\}$, where r_1, r_2 and r_3 are the reflections indicated in Figure 3. As in Example 2.7, (W, R) is not a simple system because the corresponding hyperplanes H_1, H_2 do not form a Weyl chamber. For λ_1, λ_2 real and as shown in Figure 3, the triple (W, R, Λ) is not well-framed. In fact, for every choice of a real frame $\Lambda \subset \mathbb{R}^2$, the resulting system (W, R, Λ) is not well-framed. However, it is still possible to construct a well-framed system from (W, R) using $\Lambda \subset \mathbb{C}^2$.

Let H_1, H_2, H_3 be the (complex) reflecting hyperplanes for r_1, r_2, r_3, respectively, and choose a nonzero real vector $v \in H_2$, so that $w = r_3 v$ is a nonzero real vector in H_1. Set $\lambda_1 = iv$ and $\lambda_2 = w$. Then it is straightforward to check that (W, R, Λ) is a well-framed

system. For example, the two segments

$$\sigma_1 = \Lambda_R = \{tiv + (1-t)w \ : \ 0 \leq t \leq 1\}$$

and

$$\sigma_2 = r_3\sigma_1 = \{siw + (1-s)v \ : \ 0 \leq s \leq 1\}$$

do not intersect because the equation

$$tiv + (1-t)w = siw + (1-s)v$$

has no solution with $0 \leq s, t \leq 1$, and in fact no real solution $s, t \in \mathbb{R}$ at all, since

$$(ti + s - 1)v = (si + t - 1)w$$

implies that $ti + s - 1 = si + t - 1 = 0$ by linear independence of v, w, and so $s = t = \frac{1-i}{2}$.

Note also that the linear form $\alpha_{H_3}(x) := \langle x, z \rangle$ with $z \in \mathbb{R}^2$ nonzero and perpendicular to H_3, is nowhere zero on σ_1 and σ_2, and so H_3 intersects neither σ_1 nor σ_2.
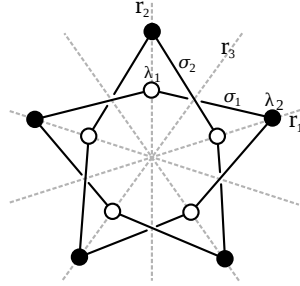

FIGURE 3. System (W, R, Λ) with $W = I_2(5)$ and $R = \{r_1, r_2\}$. The system (W, R) is well-framed for some nonreal $\Lambda = \{\lambda_1, \lambda_2\} \subset \mathbb{C}^2$, but not for any real $\Lambda \subset \mathbb{R}^2$.

2.3. **Geometry and algebra.** Although $\Delta(W, R)$ is generally only a Boolean complex, existence of a well-framed system (W, R, Λ) forces it to be a simplicial complex.

Proposition 2.9. *If (W, R, Λ) is well-framed, then $\Delta := \Delta(W, R)$ is a balanced simplicial complex.*

Proof. The image of a face under ρ is determined by its vertices. Because ρ is assumed to be an embedding, it follows that any two faces of Δ with equal vertex sets must be the same face. $\square$

We also note that every well-framed system (W, R, Λ) with W a real reflection group and $\Lambda \subset \mathbb{R}^\ell$ is of the type constructed in Example 2.5:

Proposition 2.10. *Let (W, R) be a well-generated system with W a (finite) real reflection group. Assume that $\Lambda \subset \mathbb{R}^\ell$ is a real frame. Then (W, R, Λ) is well-framed if and only if*

$$\mathbb{R}_{\geq 0}\lambda_1, \ldots, \mathbb{R}_{\geq 0}\lambda_\ell$$

are the extreme rays of a Weyl chamber. Moreover, if (W, R, Λ) is well-framed, then (W, R) is a simple system, and $\rho(\Delta)$ is isomorphic to the Coxeter complex via radial projection. $\square$

Proof. Suppose (W, R, Λ) is well-framed. The (closed) Weyl chambers are simplicial cones cut out in $\mathbb{R}^\ell$ by the reflecting hyperplanes, and the λ_i's are linearly independent and lie on extreme rays of some of the cones, so if the λ_i's are not in a single cone, then neither is their real hull Λ_R, and the interior of Λ_R meets some reflecting hyperplane of W, so that $r\Lambda_R = \Lambda_R$ for a reflection r, and hence $rW_\varnothing = W_\varnothing$, which is nonsense. The other assertions are clear. $\qquad\square$

2.4. **The support map.** The support map connects the Coxeter complex and the lattice of reflecting hyperplane intersections in the classical real case. We extend the connection to well-framed systems in the next subsection, after introducing some terminology.

Definition 2.11. Let (W, R, Λ) be well-framed. Define the *support* map to be

$$\mathrm{Supp} : \Delta \longrightarrow \mathscr{L}(V)$$
$$gW_J \longmapsto \mathrm{Span}(\rho(gW_J)),$$

and let

$$\Pi_W \overset{\mathbf{def}}{=} \{\mathrm{Supp}(\sigma) \ : \ \sigma \in \Delta\},$$

viewed as a subposet of $\mathscr{L}(V)$.

Just as for $\mathscr{L}_W$, we will usually drop W from Π_W and simply write Π. Observe that for a well-framed system one has that $\mathrm{Supp} : \Delta_W \to \mathscr{L}_W$:

Proposition 2.12. *For a well-framed system (W, R, Λ) we have*

$$\Pi_W \subseteq \mathscr{L}_W$$

with equality if and only if (W, R, Λ) is strongly stratified.

Proof. Consider a coset $gW_{R \setminus J}$, and recall that $\rho(gW_{R \setminus J}) = g\Lambda_J$. Using the definition of Λ, we also have that

$$\mathrm{Span}(\Lambda_J) = \bigcap_{r_i \in R \setminus J} H_i.$$

The inclusion follows. The second claim follows from the definition of strongly stratified system by considering dimension. $\qquad\square$

2.5. **A Galois correspondence.** The main theorem of this section is that the equivariant support map for a well-framed system has a purely algebraic description, thanks to an analogue (Theorem 2.13) of the Galois correspondence for Coxeter groups [4, Thm. 3.1].

Recall the poset of *parabolic subgroups*

$$P(W, R) \overset{\mathbf{def}}{=} \{gW_J g^{-1} \ : \ g \in W, \ \ J \subseteq R\}$$

ordered by inclusion, i.e.,

$$gW_J g^{-1} < hW_{J'} h^{-1} \quad \text{if} \quad gW_J g^{-1} \subset hW_{J'} h^{-1},$$

and note that W acts on $P(W, R)$ by conjugation.

Theorem 2.13 (Galois Correspondence)**.** *For (W, R, Λ) a well-framed system,*

$$\mathrm{Stab} : \ \Pi_W \longrightarrow P(W, R)$$
$$X \longmapsto \{g \in W \ : \ gx = x \text{ for all } x \in X\}$$

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is a W-poset isomorphism, with inverse

$$\mathrm{Fix}: \ P(W, R) \longrightarrow \Pi_W$$

$$G \longmapsto V^G := \{v \in V \ : \ gv = v \quad \textit{for all} \quad g \in G\}.$$

Proof. Use that ρ is an equivariant embedding of a balanced complex to write

$$\mathrm{Stab}(\mathrm{Supp}(gW_{R\backslash S})) = \{w \in W \ : \ w\rho(gW_{R\backslash S}) = \rho(gW_{R\backslash S}) \text{ pointwise}\}$$

$$= \{w \in W \ : \ wgW_{R\backslash S} = gW_{R\backslash S}\}$$

$$= gW_{R\backslash S}g^{-1}.$$

For the inverse, write

$$V^{g(W_{R\backslash S})g^{-1}} = gV^{W_{R\backslash S}} = g \bigcap_{r_i \in R\backslash S} H_i = g \cdot \mathrm{Span}(\Lambda_S) = \mathrm{Supp}(gW_{R\backslash S}). \qquad \square$$

The promised algebraic interpretation of support is now encoded in an equivariant commutative diagram:

Theorem 2.14. *For a well-framed system (W, R, Λ), we have the following commutative diagram of equivariant maps:*

$$
\begin{array}{ccccc}
gW_J & & \Delta & \xrightarrow{\ \underset{\rho}{\sim}\ } & \rho(\Delta) \\
\big\downarrow & & \vdots & & \big\downarrow{\scriptstyle \mathrm{Span}} \\
gW_Jg^{-1} & P(W, R) & \underset{\underset{\mathrm{Stab}}{\longleftarrow}}{\overset{\mathrm{Fix}}{\underset{\sim}{\longrightarrow}}} & \Pi_W & \overset{\iota}{\hookrightarrow} \mathscr{L}_W
\end{array}
$$

If (W, R) is strongly stratified, then $\Pi_W = \mathscr{L}_W$. $\qquad \square$

3. Pointed objects and local conicality

This section has four parts. Section 3.1 introduces the main objects of the paper: certain subcomplexes Δ_T^U of $\Delta(W, R)$ and certain subposets Π_T^U of $\mathscr{L}_W$, where U and T are subsets of the generators R.

Ehrenborg and Jung's pointed objects correspond to the special case when $W = \mathfrak{S}_n$ and $U = \{(n-1, n)\}$, and their result says that, in this case one has that for all T the top homology groups of Δ_T^U and $\Pi_T^U \setminus \{\hat{1}\}$ are isomorphic as $\mathfrak{S}_{n-1}$-modules; we elaborate on this specialization in Section 3.4, where we give a detailed dictionary between our geometric language and their combinatorial language.

In the middle two sections we prepare a much stronger and more general version of the type A_n phenomenon for use within our general framework of well-framed systems from Section 2. In Section 3.2 we isolate well-framed systems (W, R) that have a special property which guarantees that not only are the top homology modules of Δ_T^U and $\Pi_T^U \setminus \{\hat{1}\}$ isomorphic for all T and nonempty U, but in fact Δ_T^U and $\Pi_T^U \setminus \{\hat{1}\}$ are equivariantly homotopy equivalent for all such choices. We call the special systems *locally conical.*

In Section 3.3 we apply homology to the homotopy result for locally conical systems and focus on the case when $\Delta(W, R)$ is Cohen–Macaulay, so that the homology is concentrated in the top dimension and we can apply the Hopf-trace formula to write the character as an alternating sum of principal characters induced from standard parabolics W_J.

Later, in Sections 4 and 5 we show that simple systems and what we call Shephard systems are both locally conical and Cohen–Macaulay, so that all of these results apply to finite irreducible Coxeter groups and Shephard groups.

3.1. **Definitions and examples.** In this section we introduce the main objects of this paper, the generalizations of Ehrenborg and Jung's pointed objects. Recall that the *(closed) star* $\mathrm{St}_\Sigma(\sigma)$ of a simplex σ in a simplicial complex Σ is the subcomplex of all faces that are joinable to σ within Σ. That is,

$$\mathrm{St}_\Sigma(\sigma) = \{\tau \ : \ \tau \in \Sigma \quad \text{and} \quad \tau \cup \sigma \in \Sigma\}$$

where henceforth we write $\tau * \sigma$ for the join $\tau \cup \sigma$.

Definition 3.1. Let (W, R, Λ) be a well-framed system. Let $U \subseteq R$. The *subcomplex pointed by U* is

$$\Delta^U \ \overset{\mathbf{def}}{=} \ \mathrm{St}_\Delta(W_{R \setminus U}).$$

The corresponding *pointed poset of flats* is

$$\Pi^U \ \overset{\mathbf{def}}{=} \ \{\mathrm{Supp}(\sigma) \ : \ \sigma \in \Delta^U\},$$

as a subposet of $\mathscr{L}_W$.
   For $T \subseteq R$, let

$$\Delta_T^U \ \overset{\mathbf{def}}{=} \ \Delta^U \mid_T,$$

meaning $\Delta_T^U = \{\sigma \in \Delta^U \ : \ \mathrm{type}(\sigma) \subseteq T\}$ with $\mathrm{type}(gW_{R \setminus J}) = J$ as in §2.1, and let

$$\Pi_T^U \ \overset{\mathbf{def}}{=} \ \{\mathrm{Supp}(\sigma) \ : \ \sigma \in \Delta_T^U\}.$$

   Note that by Theorem 2.14, we have $\mathrm{Supp}(gW_J) = V^{gW_J g^{-1}}$, so that

$$(2) \qquad\qquad \Pi_T^U \ = \{V^{gW_J g^{-1}} \ : \ gW_J \in \Delta_T^U\}.$$

Example 3.2. Figure 4 illustrates the construction of Δ_T^U for $W = \mathfrak{S}_4$ and eight choices of T and U. We have written U above each element of U, and T below each element of T in the Coxeter diagram $\mathcal{D}$ of (W, R). Labeling the generators $R = \{r_0, r_1, \ldots, r_{\ell-1}\}$, the vertex marked with i in $\mathcal{D}$ represents the reflection r_i whose hyperplane can be written as

$$H_i = \mathrm{Span}(\Lambda \setminus \{\lambda_i\}).$$

   Recall that W is the symmetry group of the tetrahedron $\mathcal{P}$, so that the barycentric subdivision $B(\mathcal{P})$ of the boundary of $\mathcal{P}$ is homeomorphic to the Coxeter complex via radial projection. The vertex of Δ marked with i corresponds to λ_i. It happens that W is also a Shephard group because $\mathcal{P}$ is a regular polytope. In the later notation of Shephard groups, vertex i will correspond to a face B_i in a distinguished *base flag* $\mathcal{B}$ (a chamber in the flag complex $K(\mathcal{P})$ for the polytope $\mathcal{P}$).

Example 3.3. Figure 5 illustrates Δ_T^U for the hyperoctahedral group $\mathbb{Z}/2\mathbb{Z} \wr \mathfrak{S}_n$ of $n \times n$ signed permutation matrices when $n = 3$. Recall that this is the symmetry group of the n-cube $\mathcal{P}$, so its Coxeter complex is a radial projection of the barycentric subdivision $B(\mathcal{P})$
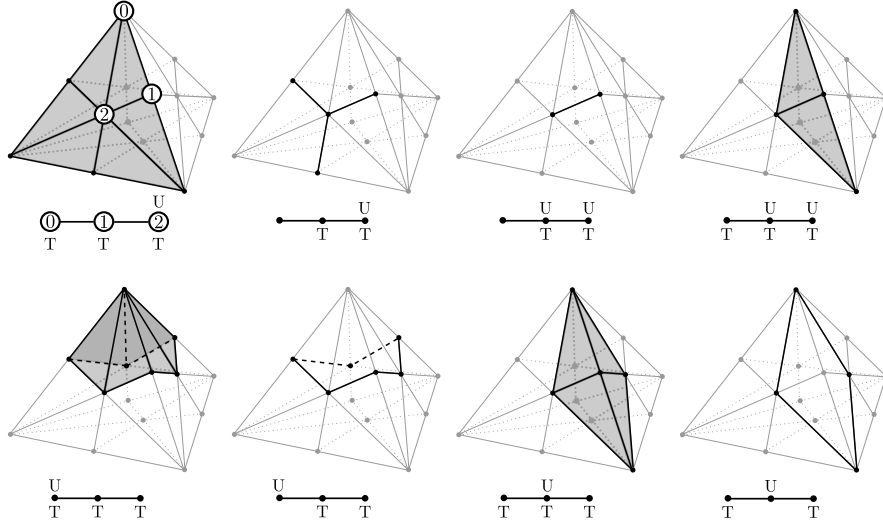
11

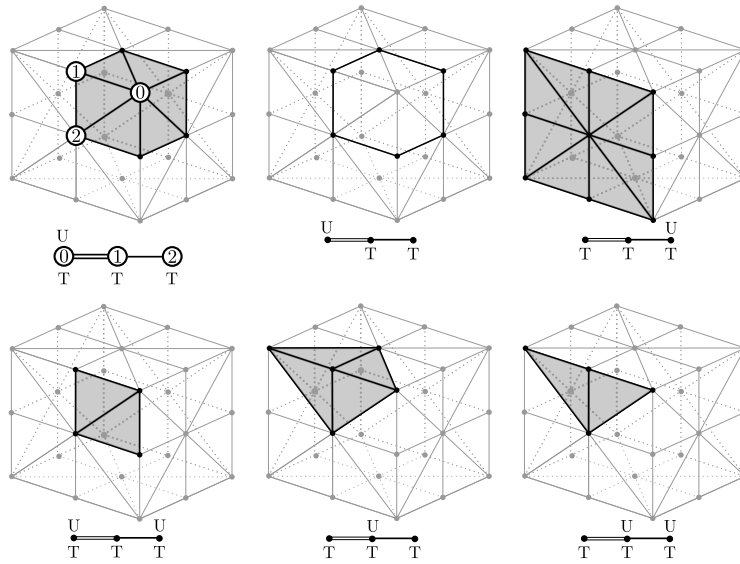FIGURE 4. A sampling of Δ_T^U for $W = \mathfrak{S}_4$.

FIGURE 5. A sampling of Δ_T^U for $W = \mathbb{Z}/2\mathbb{Z} \wr \mathfrak{S}_3$.

of the boundary of $\mathscr{P}$. Again, we let i denote our choice of λ_i, and in later notation, a face B_i of a distinguished base flag $B_0 \subset B_1 \subset \ldots \subset B_{\ell-1}$ in the flag complex $K(\mathscr{P})$ of $\mathscr{P}$.

3.2. **An equivariant homotopy for locally conical systems.** The aim of this section is to establish a sufficient condition for the order complex of $\Pi_T^U \setminus \{\hat{1}\}$ to be equivariantly homotopy equivalent to Δ_T^U.

We start by recalling some standard terminology, notation, and classical facts; see [33]. A map $f : P \to Q$ of posets is *order-preserving* if $f(p_1) \leq f(p_2)$ whenever $p_1 \leq p_2$, and *order-reversing* if $f(p_1) \geq f(p_2)$ whenever $p_1 \leq p_2$. A G-poset is a poset with a G-action

that preserves order, and a map $f : P \to Q$ of such posets is *G-equivariant* if it is a mapping of G-sets, i.e., if $f(gp) = gf(p)$ for all $g \in G$ and $p \in P$.

The order complex of a poset P is the simplicial complex of all totally ordered subsets of P, and is denoted by $\Delta(P)$. The *face poset* $\mathrm{Face}(\Sigma)$ of a simplicial complex Σ is the poset of all *nonempty* faces ordered by inclusion. Finally, $\simeq$ denotes homotopy equivalence, with added decoration to indicate equivariance. Note that the barycentric subdivision $\Delta(\mathrm{Face}(\Sigma))$ is homeomorphic to Σ.

We require a specialization[2] of Thévenaz and Webb's equivariant version of Quillen's fiber lemma; see also [33, §5.2].

Theorem 3.4 (Thévenaz and Webb [32])**.** *Let Q be a G-poset, and let Σ be a simplicial complex with a G-action. If $\phi : \mathrm{Face}(\Sigma) \to Q$ is an order-reversing G-equivariant map of G-posets such that the order complex $\Delta(\phi^{-1}(Q_{\geq q}))$ is $\mathrm{Stab}_G(q)$-contractible for all $q \in Q$, then $\Sigma \simeq_G \Delta(Q)$.*

Consider a simplicial complex Σ and order-reversing map $\phi : \mathrm{Face}(\Sigma) \to Q$ of posets. Since $\phi^{-1}(Q_{\geq q})$ is an order ideal in $\mathrm{Face}(\Sigma)$, it is the face poset $\mathrm{Face}(\Phi)$ of some subcomplex $\Phi \subseteq \Sigma$. Call such a subcomplex Φ a *Quillen fiber*. Note that $\Delta(\phi^{-1}(Q_{\geq q}))$ is homeomorphic to Φ, so that Quillen's fiber lemma concerns contractibility of Quillen fibers.

The following definition of *locally conical system* is central to our work. We will show that for such systems, Theorem 3.4 can be applied to establish the desired homotopy equivalence between $\Pi_T^U \setminus \{\hat{1}\}$ and Δ_T^U.

Definition 3.5. A well-framed system (W, R, Λ) is *locally conical* if for each nonempty $U \subseteq R$, every Quillen fiber
$$X \cap \Delta^U \qquad (X \in \Pi^U \setminus \{\hat{1}\})$$
of $\mathrm{Supp} : \mathrm{Face}(\Delta^U) \to \Pi^U \setminus \{\hat{1}\}$ has a cone point.

Recall that a *cone point* p of a simplicial complex Σ is a vertex of Σ that is joinable in Σ to every simplex of Σ, i.e., every maximal simplex of Σ contains p.

Proposition 3.6. *Let Σ be a simplicial complex with a G-action. If Σ has a cone point, then Σ is G-contractible.*

Proof. The union of all cone points must form a G-stable simplex of Σ, whose barycenter p is therefore a G-fixed point of the geometric realization $\|\Sigma\|$. Since p lies in a common simplex with every simplex of Σ, this space $\|\Sigma\|$ is star-shaped with respect to the G-fixed point p, and a straight-line homotopy retracts $\|\Sigma\|$ onto p in a G-equivariant fashion. $\qquad\square$

Before employing Theorem 3.4, recall that $\Delta^U = \mathrm{St}_\Delta(W_{R \setminus U})$ and that the action of W on Δ preserves types. Hence, the subcomplex Δ_T^U is a $W_{R \setminus U}$-poset. It follows that
$$\mathrm{Supp} : \mathrm{Face}(\Delta_T^U) \to \Pi_T^U \setminus \{\hat{1}\}$$
is an order-reversing $W_{R \setminus U}$-equivariant map.

---

[2]The main theorem of [32] states that $\Delta(P) \simeq_G \Delta(Q)$ if $\phi : P \to Q$ is an order-preserving G-equivariant map of G-posets such that each fiber $\Delta(\phi^{-1}(Q_{\leq q}))$ is $\mathrm{Stab}_G(q)$-contractible. Theorem 3.4 specializes P to $\mathrm{Face}(\Sigma)$ and replaces Q with its *opposite* Q^{opp}, using the fact that $\Sigma \simeq_G \Delta(\mathrm{Face}(\Sigma))$ and $\Delta(Q^{\mathrm{opp}}) = \Delta(Q)$. Recall that Q^{opp} is obtained from Q by reversing order.

Theorem 3.7. *Let (W, R, Λ) be a locally conical system. Let $U \subseteq R$ be nonempty and $T \subseteq R$. Then $\Delta(\Pi_T^U \setminus \{\hat{1}\})$ is $W_{R \setminus U}$-homotopy equivalent to Δ_T^U.*

Proof. We apply Theorem 3.4 to the map

$$\mathrm{Supp} : \mathrm{Face}(\Delta_T^U) \twoheadrightarrow \Pi_T^U \setminus \{\hat{1}\}.$$

Let $X \in \Pi_T^U \setminus \{\hat{1}\} \subseteq \Pi^U \setminus \{\hat{1}\}$, and consider first the Quillen fiber

$$(3) \qquad\qquad\qquad\qquad \Phi := X \cap \Delta^U$$

for the unrestricted map $\mathrm{Supp} : \mathrm{Face}(\Delta^U) \to \Pi^U \setminus \{\hat{1}\}$.

By definition of locally conical system, Φ has a cone point p. Since Δ is balanced, the subcomplex Φ is balanced as well, and so p is the unique vertex of Φ of its type.

Choose $\sigma \in \Delta_T^U$ with $X = \mathrm{Supp}(\sigma)$. By the construction of $\rho(\Delta)$, the vertices of the join $\{p\} * \sigma$ are contained in

$$g\Lambda = \{g\lambda_1, g\lambda_2, \ldots, g\lambda_\ell\}$$

for some $g \in W$. Because $g\Lambda$ is a linearly independent set, $p \in \mathrm{Supp}(\sigma)$ implies that p is a vertex of σ, and hence a vertex of Δ_T^U. Therefore, the restricted Quillen fiber $X \cap \Delta_T^U$ also has p as a cone point. It follows from Proposition 3.6 that $X \cap \Delta_T^U$ is $\mathrm{Stab}_{W_{R \setminus U}}(X)$-contractible. $\qquad\square$

3.3. **Homology of locally conical systems.** Applying the homology functor to the homotopy equivalence in Theorem 3.7 gives the following theorem.

Theorem 3.8. *Let (W, R, Λ) be a locally conical system. Let $U \subseteq R$ be nonempty and $T \subseteq R$. Then $\bigoplus_i \widetilde{H}_i(\Delta_T^U)$ and $\bigoplus_i \widetilde{H}_i(\Delta(\Pi_T^U \setminus \{\hat{1}\}))$ are isomorphic as graded $(W_{R \setminus U})$-modules.*

This section is devoted to establishing explicit descriptions for the modules in Theorem 3.8 when Δ is Cohen–Macaulay. Recall that a simplicial complex Σ is *Cohen–Macaulay* (over $\mathbb{Z}$) if for each $\sigma \in \Sigma$ we have $\widetilde{H}_i(\mathrm{lk}\, \sigma, \mathbb{Z}) = 0$ whenever $i < \dim(\mathrm{lk}_\Sigma \sigma)$, where lk_Σ denotes the *link*:

$$\mathrm{lk}_\Sigma(\sigma) = \{\tau \in \Sigma \; : \; \tau \cap \sigma = \varnothing, \; \tau \cup \sigma \in \Sigma\}.$$

The complex Σ is *homotopy Cohen–Macaulay* if for each $\sigma \in \Sigma$ we have $\pi_r(\mathrm{lk}_\Sigma(\sigma)) = 0$ whenever $r \leq \dim \mathrm{lk}_\Sigma(\sigma) - 1$. Homotopy Cohen–Macaulay implies Cohen–Macaulay (over $\mathbb{Z}$), and Cohen–Macaulay implies Cohen–Macaulay over any field; see the appendix of [7].

We start with the following proposition.

Proposition 3.9. *Let (W, R, Λ) be a well-framed system. Let $U, T \subseteq R$. If Δ is Cohen–Macaulay (resp. homotopy Cohen–Macaulay), then Δ_T^U is Cohen–Macaulay (resp. homotopy Cohen–Macaulay).*

Proof. It is an easy exercise to show that stars inherit the Cohen–Macaulay (resp. homotopy Cohen–Macaulay) property. The nontrivial step is concluding that Δ_T^U is Cohen–Macaulay (resp. homotopy Cohen–Macaulay). This follows from a type-selection theorem for pure simplicial complexes; see [8, Thm. 11.13] or [10, Thm. 4.3]. $\qquad\square$

Lemma 3.10. *Let (W, R, Λ) be a well-framed system. If $U \cap T \neq \varnothing$, then Δ_T^U is contractible.*

Proof. This is immediate from $\Delta^U = \{W_{R \setminus U}\} * \mathrm{lk}_\Delta W_{R \setminus U}$. $\qquad\square$

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Theorem 3.11. *Let (W, R, Λ) be a well-framed system. Let $U, T \subseteq R$, and assume that Δ is Cohen–Macaulay. If $U \cap T \neq \varnothing$, then the top homology $\widetilde{H}_{|T|-1}(\Delta_T^U)$ is trivial; otherwise,*

$$(4) \qquad \widetilde{H}_{|T|-1}(\Delta_T^U) \cong \sum_{J \subseteq T} (-1)^{|T \setminus J|} \mathrm{Ind}_{W_{R \setminus (U \cup J)}}^{W_{R \setminus U}} \mathbb{1}$$

as virtual $(W_{R \setminus U})$-modules.

Remark 3.12. Specializing to type A and to Ehrenborg and Jung's objects (see Section 3.4), Lemma 3.10 translates to $\Delta_{\vec{c}}$ being contractible whenever $\vec{c}$ ends with a zero; this is precisely Lemma 3.1 in [15]. The condition that $U \cap T = \varnothing$ collapses to the condition that $\vec{c}$ does not end with zero.

We also note that the virtual modules in (4) are well known to be the natural generalization of ribbon representations to all Coxeter and Shephard groups; see [26] and [22].

Though the proof of Theorem 3.11 is entirely standard, we first need a particular description of Δ_T^U when $U \cap T = \varnothing$. The following intermediate description is straightforward.

Lemma 3.13. *Let (W, R) be a well-generated system. Then for $U, T \subseteq R$ we have*

$$(5) \qquad \Delta_T^U = \{g W_{R \setminus J} \ : \ g \in W_{R \setminus U}, \ J \subseteq T\}.$$

When $\Delta(W, R)$ is a simplicial complex, one has that (W, R) satisfies the *intersection condition*

$$(6) \qquad \bigcap_{r \in R \setminus J} W_{R \setminus \{r\}} = W_J \quad \text{for all } J \subseteq R.$$

In fact, satisfying the intersection condition is equivalent to $\Delta(W, R)$ being a simplicial complex; see [3, Cor. 2.6]. The next lemma is a straightforward application of the intersection property, and the desired description of Δ_T^U is obtained by applying the lemma to (5).

Lemma 3.14. *Let (W, R) be well-framed. If $U \cap T = \varnothing$, then the map*

$$\{g W_{R \setminus J} \ : \ g \in W_{R \setminus U}, \ J \subseteq T\} \longrightarrow \{g W_{R \setminus (U \cup J)} \ : \ g \in W_{R \setminus U}, \ J \subseteq T\}$$
$$g W_{R \setminus J} \longmapsto g W_{R \setminus (U \cup J)}$$

is a $(W_{R \setminus U})$-poset isomorphism. $\qquad\qquad\square$

The proof of Theorem 3.11 now follows:

Proof of Theorem 3.11. The first claim follows from Lemma 3.10. When $U \cap T = \varnothing$, the result follows from the description of Δ_T^U obtained through Lemmas 3.13 and 3.14 by applying the standard argument using Cohen–Macaulayness and the Hopf trace formula, as is detailed by Solomon in [27, §4]. See also [28, Thm. 1.1]. $\qquad\qquad\square$

3.4. Specializing to type A: two new proofs of Ehrenborg and Jung's result.
In the case of type A, Ehrenborg and Jung constructed *pointed objects* $\Delta_n, \Pi_n^\bullet$ and subobjects $\Delta_{\vec{c}}, \Pi_{\vec{c}}^\bullet$ indexed by compositions $\vec{c} = (c_1, c_2, \ldots, c_k)$ of n that have positive entries $c_1, \ldots, c_{k-1} > 0$ but *nonnegative* last entry $c_k \geq 0$. Figure 6 shows Ehrenborg and Jung's Δ_2 and $\Pi_2^\bullet$, each carrying an action of $\mathfrak{S}_2$. In [15], they show that the top homology of $\Delta_{\vec{c}}$ is homotopy equivalent to the order complex of $\Pi_{\vec{c}}^\bullet \setminus \{\hat{1}\}$, and that the top homology is given by a ribbon Specht module of $\mathfrak{S}_n$ with row sizes determined by $\vec{c}$.
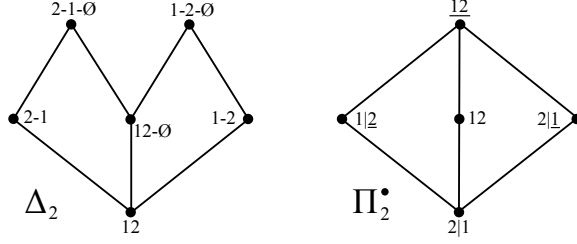
FIGURE 6. Ehrenborg and Jung's pointed objects.

The aim of this section is to present the underlying geometry of Ehrenborg and Jung's objects by showing how our objects Δ_T^U, Π_T^U specialize to theirs. It will follow that applying our results to this specialization recovers their main results, even upgrading their homotopy equivalence to an equivariant one. The translations between objects of this section are well known, and we will largely follow the discussion of Aguiar and Mahajan; see [2, §1.4].

Let $W = \mathfrak{S}_{n+1}$ and let $R = \{r_1, r_2, \ldots, r_n\}$ be the usual generating set of adjacent transpositions, i.e.,

$$r_i = (i, i+1).$$

The symmetric group $\mathfrak{S}_{n+1}$ is the symmetry group of the standard n-simplex $\mathscr{P}_n$ with vertices labeled with $\{1, 2, \ldots, n+1\}$. The barycentric subdivision $B(\mathscr{P}_n)$ of the boundary of $\mathscr{P}_n$ is homeomorphic to the Coxeter complex via radial projection. Letting λ_i be the vertex of $B(\mathscr{P}_n)$ indexed by $\{1, 2, \ldots, i\}$ gives a well-framed system (W, R, Λ) with $\rho(\Delta) = B(\mathscr{P}_n)$. Note that in particular Λ lies on the *distinguished face* $\mathscr{P}_{n-1}$ of $\mathscr{P}_n$ that has vertex set $\{1, 2, \ldots, n\}$; see Figure 7. We call (W, R, Λ) the *standard system for* $\mathfrak{S}_{n+1}$.
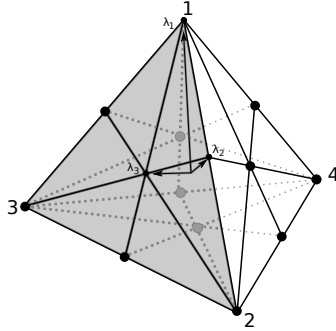

FIGURE 7. $W = \mathfrak{S}_4$ with the subdivision of facet $\mathscr{P}_2$ shaded.

A *set composition* of $n+1$ is an ordered partition B_1—B_2—$\ldots$—B_k of $[n+1]$ with nonempty blocks. The collection of all set compositions of $n+1$ forms a simplicial complex Σ_{n+1} under refinement, with facets having $n+1$ singleton blocks; see Σ_3 in Figure 9. The *type* of a set composition is the composition of its block sizes, i.e.,

$$\text{type}(B_1\text{—}B_2\text{—}\ldots\text{—}B_k) = (|B_1|, |B_2|, \ldots, |B_k|).$$

Given a composition $\vec{c}$ of $n+1$, the subcomplex of Σ_{n+1} generated by the faces of type $\vec{c}$ is denoted by $\Sigma_{\vec{c}}$.

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The map ϕ obtained by letting

$$\{\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k}\} \quad \mapsto \quad \{1, \ldots, i_1\}\text{---}\{i_1+1, \ldots, i_2\}\text{---}\ldots\text{---}\{i_k+1, \ldots, n+1\}$$

and extending by the action of W, is an equivariant isomorphism $\rho(\Delta) \to \Sigma_{n+1}$; see Figure 8 and Σ_3 in Figure 9. Under this isomorphism, a face $B_1\text{---}B_2\text{---}\ldots\text{---}B_k$ of type $\vec{c}$ corresponds to a face of type

$$\mathrm{Des}(\vec{c}) := \{r_{|B_1|}, r_{|B_1|+|B_2|}, \ldots, r_{|B_1|+\ldots+|B_{k-1}|}\},$$

the *descent set* associated with $\vec{c}$. Note also that those faces of $B(\mathscr{P}_n)$ in the star of $\mathscr{P}_{n-1}$ either have $n+1$ contained in their last block or have last block equal to $\{n+1\}$.
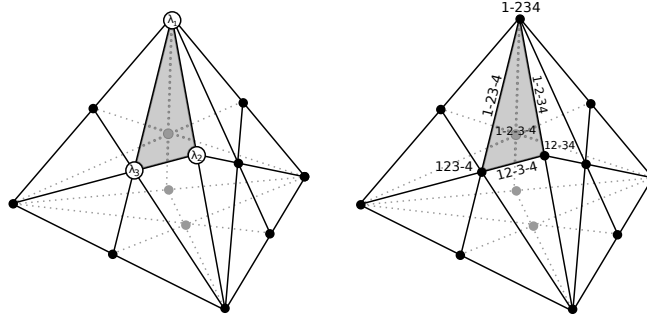

FIGURE 8. The chamber Λ_R with vertices $\lambda_1, \lambda_2, \lambda_3$ (left) and its image under ϕ (right) in the case when $W = \mathfrak{S}_4$.

The lattice of hyperplane intersections $\mathscr{L}_W$ for W also has a simple description obtained from set compositions. Under the identification of $\rho(\Delta)$ and Σ_{n+1}, the support map corresponds to forgetting the order on the blocks, so that

$$\mathrm{Supp}(B_1\text{---}B_2\text{---}\ldots\text{---}B_k) = B_1|B_2|\ldots|B_k,$$

and the induced partial order is given by refinement.

A *pointed set composition* of n is an ordered partition $B_1\text{---}B_2\text{---}\ldots\text{---}B_k$ of $[n]$ with last block B_k possibly empty. We denote the collection of all pointed set compositions of $[n]$ by $\Delta_n^\bullet$. The previous discussion shows that by removing $n+1$ from blocks in elements of Σ_{n+1}, the barycentric subdivision of the facet $\mathscr{P}_{n-1}$ can be identified with $\Delta_n^\bullet$; see Figures 8 and 10. More accurately, $\Delta_n^\bullet$ is $\mathfrak{S}_n$-equivariantly isomorphic to $\Delta^{\{r_n\}}$. Given a composition $\vec{c}$ of n with last part possibly 0, the corresponding selected complex is denoted by $\Delta_{\vec{c}}^\bullet$, which is Ehrenborg and Jung's complex $\Delta_{\vec{c}}$; see Figure 10. By distinguishing terminal blocks before taking the image of $\Delta_{\vec{c}}$ under the support map, one obtains their pointed poset $\Pi_{\vec{c}}^\bullet$ after removing any (possibly distinguished) empty blocks; see Figure 9.

Ehrenborg and Jung distinguish a block by underlining it. Thus, the above map is

$$\Delta_{\vec{c}} \longrightarrow \Pi_{\vec{c}}^\bullet$$
$$B_1\text{---}B_2\text{---}\ldots\text{---}B_k \longmapsto B_1|B_2|\ldots|\underline{B_k}.$$

The following proposition summarizes the correspondence outlined above.

Proposition 3.15. *Let (W, R, Λ) be the standard system for $\mathfrak{S}_{n+1}$, and let $\vec{c} = (c_1, c_2, \ldots, c_k)$ be a composition of n with last part c_k possibly 0. Then the following diagram composed of*
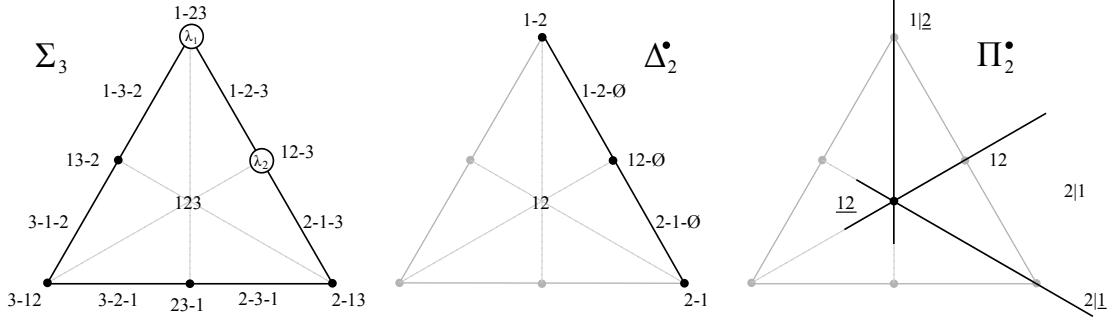
FIGURE 9. The geometry of Ehrenborg and Jung's Δ_2 and $\Pi_2^\bullet$ from Figure 6.
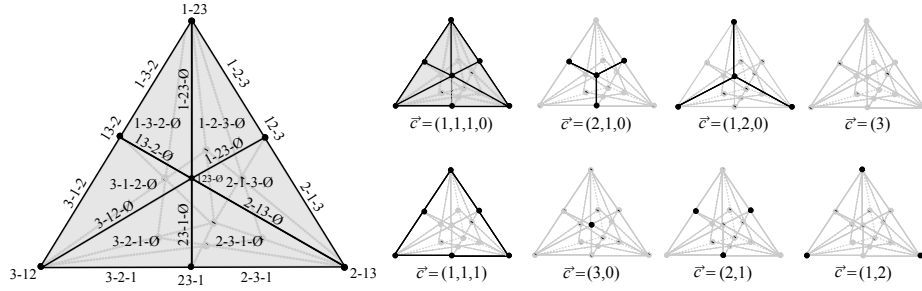

FIGURE 10. Ehrenborg and Jung's $\Delta_{\vec{c}}$ for all possible choices of $\vec{c} \vdash 3$ (with last part is allowed to be 0).

$\mathfrak{S}_n = W_{R \setminus \{r_n\}}$-equivariant maps is commutative:

The main result of [15] is the following theorem.

Theorem 3.16 (Ehrenborg and Jung). *Let $\vec{c} = (c_1, c_2, \ldots, c_k)$ be a composition of $n+1$ with last part c_k possibly 0. Then we have the following isomorphism of top (reduced) homology groups as $\mathfrak{S}_n$-modules:*

$$\widetilde{H}_{\text{top}}(\Delta(\Pi_{\vec{c}}^\bullet \setminus \{\hat{1}\})) \simeq_{\mathfrak{S}_n} \widetilde{H}_{\text{top}}(\Delta_{\vec{c}}).$$

Remark 3.17. Proposition 3.15 shows that Theorem 3.16 is implied by combining Theorem 4.1(iv) or Theorem 5.3(iv) below with Theorem 3.7 (or Theorem 3.8).

4. COXETER GROUPS

The main theorem of this section tells us that all of our results about locally conical systems from Section 3 apply to finite irreducible Coxeter groups:

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Theorem 4.1. *Let (W, R) be a simple system for a finite irreducible real reflection group W, and let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_\ell\}$ consist of one nonzero point from each extreme ray of a Weyl chamber corresponding to R. Then the following hold:*

   (i) (W, R, Λ) is strongly stratified.
   (ii) $\rho(\Delta)$ is homeomorphic to the Coxeter complex of (W, R) via radial projection.
   (iii) $\Delta := \Delta(W, R)$ is homotopy Cohen–Macaulay.
   (iv) (W, R, Λ) is locally conical.

In particular, Theorem 3.7, Theorem 3.8, and Theorem 3.11 apply to all simple systems.

The first three properties of Theorem 4.1 are clear; for (i) and (ii), recall that the reflecting hyperplanes for W cut out simplicial cones in $\mathbb{R}^\ell$ which intersect the unit sphere to form the Coxeter complex, and for (iii), the Coxeter complex in fact has the stronger property of being *shellable* [5, Thm. 2.1]; see [33, §4.1]. Property (iv) is the object of the next subsection.

4.1. **Iteratively detecting cone points.** We require the following lemma.

Lemma 4.2. *Let W be a finite irreducible real reflection group, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be nonzero vectors on extreme rays of a fixed Weyl chamber. Then the orthogonal projection of λ_1 onto $\mathrm{Span}(\lambda_2, \lambda_3, \ldots, \lambda_k)$ is nonzero.*

Proof. This follows from the claim that $\langle \lambda_i, \lambda_j \rangle > 0$ for all i, j.

To see this claim, let $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ be the simple system of roots associated with the fixed Weyl chamber C, so that α_i is orthogonal to

$$H_i = \mathrm{Span}(\lambda_1, \ldots, \hat{\lambda}_i, \ldots, \lambda_\ell),$$

and its direction is such that $\langle \lambda_i, \alpha_i \rangle \geq 0$. Recall then [11, Ch. V, §3, no. 4, Prop. 3] that the α_i form an obtuse basis for V, i.e., a basis with

$$\langle \alpha_i, \alpha_j \rangle \leq 0 \quad \forall i, j.$$

It follows from [11, Ch.V, §3, no. 5, Lemma 6] that $C \subseteq \{\Sigma_i c_i \alpha_i \; : \; c_i \geq 0\}$. This implies the weak inequality $\langle \lambda_i, \lambda_j \rangle \geq 0$ for all i, j.

The desired strict inequality then follows from the connectivity of the Coxeter diagram for W and the fact that the α_i are obtuse; see [17, p. 72, no. 8] for an outline. $\square$

We will also need some basic facts regarding convexity of the Coxeter complex; see [1], particularly Section 3.6, for details and a more general treatment. Let $W \leq \mathrm{GL}(\mathbb{R}^\ell)$ be a finite real reflection group, and let Σ denote its Coxeter complex. Recall that Σ is a *chamber complex*, meaning that all maximal simplices (called *chambers*) are of the same dimension, and any two chambers can be connected by a gallery. Here, a *gallery* connecting two chambers C, D is a sequence of chambers

$$C = C_0, C_1, \ldots, C_k = D$$

with the additional property that consecutive chambers are adjacent in the sense that they share a codimension-1 face.

A *root*[3] of Σ is the intersection of Σ with a closed half-space determined by a reflecting hyperplane, and a subcomplex of Σ is called *convex* if it is an intersection of roots. Each convex subcomplex Σ' is itself a chamber complex in which any two maximal simplices can be connected by a Σ'-gallery. Moreover, a chamber subcomplex Σ' is convex if and only if any shortest Σ-gallery connecting two chambers of Σ' is contained in Σ'.

The main tool for proving Theorem 4.1(iv) is an iterative method for detecting cone points. The following discussion and lemma make this precise.

Choose a nontrivial subset $U \subseteq R$ and consider a nontrivial simplex $\sigma \in \Delta^U$. Choose B to be a chamber of Δ^U containing $\sigma * \Lambda_U$. From a sequence $b_0, b_1, \ldots, b_k$ of distinct vertices of B, we construct a descending sequence of convex subcomplexes

$$\Delta^U = \Delta_0 \supset \Delta_1 \supset \ldots \supset \Delta_k \supset \Delta_{k+1},$$

where we set $H_i^B = \mathrm{Supp}(B \setminus \{b_i\})$ and define

$$\Delta_i = \bigcap_{j=0}^{i-1} H_j^B \cap \Delta^U.$$

We call the sequence $b_0, b_1, \ldots, b_k$ *cone-approximating* for the triple (Δ^U, σ, B) if the following two conditions are satisfied:

(1) b_i is a cone point of Δ_i for $0 \leq i \leq k$.
(2) $\sigma \in \Delta_i$ for $0 \leq i \leq k$.

Note that $\Delta_0 = \Delta^U$ implies the existence of cone-approximating sequences, since U is nontrivial. The main result is that any cone-approximating sequence can be extended to contain a vertex of σ:

Lemma 4.3. *In the above setting, a maximal cone-approximating sequence $b_0, b_1, \ldots, b_m$ for (Δ^U, σ, B) has $b_m \in \sigma$.*

Proof. Suppose that $b_m \notin \sigma$, so that $\sigma \in \mathrm{lk}_{\Delta_m}(b_m)$. Then

$$\sigma \in H_m^B \cap \Delta_m = \Delta_{m+1}.$$

Note that Δ_{m+1} is a convex subcomplex. Set

$$B_{m+1} = B \setminus \{b_0, \ldots, b_m\},$$

the distinguished chamber of Δ_{m+1} containing σ, and let $\widetilde{B}_{m+1}$ be another chamber of Δ_{m+1}.

Since Δ_{m+1} is convex, we have that

$$\Delta_{m+1} * \{b_0, \ldots, b_m\}$$

is a convex chamber subcomplex of Δ^U. Consider a gallery in $\Delta_{m+1} * \{b_0, \ldots, b_m\}$ that connects chambers $B_{m+1} * \{b_0, \ldots, b_m\}$ and $\widetilde{B}_{m+1} * \{b_0, \ldots, b_m\}$, and has no two consecutive chambers being equal. Write $\bar{\mathbf{r}}$ for the associated sequence of reflections, so that $\bar{\mathbf{r}}$ induces a gallery from B_{m+1} to $\widetilde{B}_{m+1}$ in Δ_{m+1}.

Any two chambers of $\Delta_0 = \Delta^U$ have equal support (namely the entire space V), and from the equation $\Delta_{i+1} = H_i^B \cap \Delta_i$ it follows that the same is true for all other Δ_i's, so

---

[3]Root vectors of W in $\mathbb{R}^\ell$ (vectors perpendicular to reflecting hyperplanes) are in canonical 1-1 correspondence with roots of Σ. The notion of root extends to arbitrary thin chamber complexes by introducing the notion of a folding. The terminology is due to Tits, who characterized abstract Coxeter complexes as precisely those thin chamber complexes with "enough" foldings; see [1, Sec. 3.4].

that in particular, if a reflection brings some chamber of Δ_{m+1} to another chamber of Δ_{m+1}, then the reflection stabilizes $\mathrm{Supp}(\Delta_{m+1})$. Hence $\bar{\mathbf{r}}$ stabilizes $\mathrm{Supp}(\Delta_{m+1})$ in the sense that each reflection in $\bar{\mathbf{r}}$ stabilizes $\mathrm{Supp}(\Delta_{m+1})$, which implies that $\bar{\mathbf{r}}$ stabilizes the orthogonal complement of $\mathrm{Supp}(\Delta_{m+1})$ as well. Because a subspace is stabilized by a reflection only if the subspace or its orthogonal complement is fixed pointwise by the reflection, the sequence $\bar{\mathbf{r}}$ in fact fixes the orthogonal complement of $\mathrm{Supp}(\Delta_{m+1})$ pointwise, and because b_m is fixed by $\bar{\mathbf{r}}$ as well, it follows that the orthogonal projection $\pi(b_m)$ of the vector b_m onto $\mathrm{Supp}(\Delta_{m+1})$ is fixed by $\bar{\mathbf{r}}$.

The orthogonal projection $\pi(b_m)$ is nonzero by Lemma 4.2, so that

$$\dim\ \pi(b_m) = 1,$$

and in fact $\pi(b_m)$ has nontrivial intersection with the cone over B_{m+1}, so that

$$(7) \qquad\qquad \dim\ \pi(b_m) \cap \mathbb{R}_{>0} B_{m+1} = 1,$$

because pairs of walls in a chamber do not intersect obtusely; indeed, writing s_i, s_j for the reflections in two distinct walls of a chamber, we have that $s_i \neq s_j$ and the dihedral angle formed by the walls is π/m_{ij}, where m_{ij} is the order of $s_i s_j$.

Equation (7) tells us that the line $L = \mathrm{Span}\ \pi(b_m)$ intersects a face F of B_{m+1} that is minimal in the sense that no proper face of F meets L. Because the line L is fixed by each reflection in $\bar{\mathbf{r}}$, the face F is fixed pointwise by $\bar{\mathbf{r}}$ as well. Letting b_{m+1} be a vertex of F, we conclude that any gallery in Δ_{m+1} that contains B_{m+1} has b_{m+1} as a cone point, and so by connectivity Δ_{m+1} has b_{m+1} as a cone point. Since $\sigma \in \Delta_{m+1}$ and $b_{m+1} \neq b_0, \ldots, b_m$, we can therefore append b_{m+1} to get a longer approximating sequence. $\qquad\square$

Proof of Theorem 4.1(iv). Let $U \subseteq R$ be nonempty and $\sigma \in \Delta^U \setminus \{\varnothing\}$. Choose B to be a chamber of Δ^U containing $\sigma * \Lambda_U$. Consider the Quillen fiber $\mathrm{Supp}(\sigma) \cap \Delta^U$ for the map $\mathrm{Supp} : \mathrm{Face}(\Delta^U) \to \Pi^U \setminus \{\hat{1}\}$, as in (3) of Theorem 3.7, and let $b_0, b_1, \ldots, b_m$ be a maximal cone-approximating sequence for (Δ^U, σ, B). By Lemma 4.3, Δ_m has cone point b_m that is also a vertex of σ. We want to show that b_m is also a cone point for the Quillen fiber.

Since $\sigma \in \Delta_m$ and $\Delta_m = X \cap \Delta^U$ for some particular $X \in \mathscr{L}_W$, it follows that $\sigma \subset X$, and hence

$$\mathrm{Supp}(\sigma) \cap \Delta^U \subseteq \Delta_m.$$

Let $\tau \in \mathrm{Supp}(\sigma) \cap \Delta^U$. We want to show that the join $\tau * \{b_m\}$ is also in the fiber. Since $\tau * \{b_m\} \in \Delta_m \subseteq \Delta^U$, we need only show that $\mathrm{Supp}(\tau * \{b_m\}) \subseteq \mathrm{Supp}(\sigma)$. But this is clear, since $\tau \subset \mathrm{Supp}(\sigma)$ and $b_m \in \sigma$. $\qquad\square$

5. Shephard groups

In this section we give the analogue of Theorem 4.1 for another large family of finite irreducible reflection groups called Shephard groups. The analogue is Theorem 5.3 below, and the upshot will be that all previous results of Section 3 apply not only to finite irreducible Coxeter groups, but to Shephard groups as well.

5.1. **Preliminaries.** *Shephard groups* form an important class of complex reflection groups. They are the symmetry groups of regular complex polytopes, as defined by Shephard [24]. Here we will follow Coxeter's treatment [14].

Let $\mathscr{P}$ be a finite arrangement of complex affine subspaces of V, with partial order given by inclusion. We call its elements *faces* and denote an i-dimensional face by F_i. A 0-dimensional

face is called a *vertex*. Allowing *trivial faces* $F_\ell = V$ and $F_{-1} = \varnothing$, all other faces are called *proper faces*. A totally ordered set of proper faces is called a *flag*. The simplicial complex of all flags is called the *flag complex* and is denoted by $K(\mathscr{P})$. This is the order complex of $\mathscr{P}$ with its improper faces $\varnothing$ and V omitted.

Such an arrangement $\mathscr{P}$ is a *(complex) polytope* if the following hold:

(i) $\varnothing, V \in \mathscr{P}$.
(ii) If $F_i \subset F_j$ and $|i - j| \geq 3$, then the open interval

$$(F_i, F_j) = \{F \in \mathscr{P} \ : \ F_i \subsetneqq F \subsetneqq F_j\}$$

is connected in the sense that its Hasse diagram is a connected graph.
(iii) If $F_i \subset F_j$ and $|i - j| \geq 2$, then the open interval (F_i, F_j) contains at least two distinct k-dimensional faces F_k, F_k' for each k with $i < k < j$.

For $\mathscr{P}$ a polytope, note that properties (i) and (iii) enable one to extend any partial flag $F_{i_1} \subset F_{i_2} \subset \ldots \subset F_{i_k}$ in $K(\mathscr{P})$ to a maximal flag (under inclusion) of the form

$$F_0 \subset F_1 \subset \ldots \subset F_{\ell-1}.$$

We call maximal flags in $K(\mathscr{P})$ *chambers*. If the group $W \subset \mathrm{GL}(V)$ of automorphisms of $\mathscr{P}$ acts transitively on the chambers of $\mathscr{P}$, then we say that $\mathscr{P}$ is *regular* and that W is a *Shephard group*.

The complexifications of the two (affine) arrangements shown in Figure 11 are examples of regular (complex) polygons. Both polygons have symmetry group $I_2(5)$, the dihedral group of order 10.
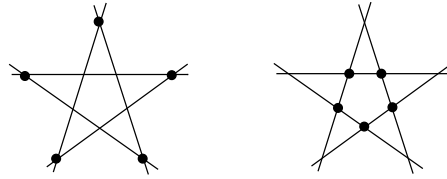

FIGURE 11. Two regular polygons with symmetry group $I_2(5)$.

If $\mathscr{P}$ contains a pair of distinct vertices that are at the minimum distance apart (among all pairs of distinct vertices) with no edge of $\mathscr{P}$ connecting them, then $\mathscr{P}$ is *starry*; see [24, p. 87]. For example, the first polytope of Figure 11 is *starry*, whereas the second is *nonstarry*. From the work of Coxeter, each Shephard group W is the symmetry group of two (possibly isomorphic) nonstarry regular complex polytopes; see Tables IV and V in [14]. Henceforth, we assume that all polytopes are nonstarry. The importance of this assumption will become clear in Theorem 5.3.

Given a regular complex polytope $\mathscr{P}$ and a choice of maximal flag

$$\mathcal{B} = B_0 \subset B_1 \subset \ldots \subset B_{\ell-1},$$

called the *base chamber*, for each i the group

$$\mathrm{Stab}_W(B_0 \subset \ldots \subset \hat{B}_i \subset \ldots \subset B_{\ell-1})$$

is generated by some reflection r_i. Choosing such an r_i for each i produces an associated set $R = \{r_0, r_1, \ldots, r_{\ell-1}\}$ that generates W. We call R a set of *distinguished generators*.

In the case when $\mathscr{P}$ has a real form, each r_i is uniquely determined, and they give the usual Coxeter presentation for W. In general, Coxeter shows that one can always choose the reflections r_i so that for some integers $p_0, p_1, \ldots, p_{\ell-1}, q_0, q_1, \ldots, q_{\ell-2}$ the group has the following Coxeter-like presentation with defining relations

$$r_i^{p_i} = 1$$
$$r_i r_j = r_j r_i, \quad |i - j| \geq 2$$
$$\underbrace{r_i r_{i+1} r_i \ldots}_{q_i} = \underbrace{r_{i+1} r_i r_{i+1} \ldots}_{q_i}.$$

These relations are encoded by *symbol*

$$p_0[q_0]p_1[q_1]p_2 \ldots p_{\ell-2}[q_{\ell-2}]p_{\ell-1},$$

which is uniquely determined by W up to reversal. The corresponding nonstarry polytopes are denoted by

$$p_0\{q_0\}p_1\{q_1\}p_2 \ldots p_{\ell-2}\{q_{\ell-2}\}p_{\ell-1} \quad \text{and} \quad p_{\ell-1}\{q_{\ell-2}\}p_{\ell-2} \ldots p_2\{q_1\}p_1\{q_0\}p_0,$$

the second called the *dual* of the first. If we denote one of the two polytopes by $\mathscr{P}$, then the other is denoted by $\mathscr{P}^*$. The two polytopes have dual face posets and isomorphic flag complexes.

The complete classification of Shephard groups is quite short:

- The symmetry groups of real regular polytopes, i.e., the Coxeter groups with connected unbranched diagrams: types $A_n, B_n = C_n, F_4, H_3, H_4, I_2(n)$.
- $p_0[q]p_1$ with $p_0, p_1 \geq 2$ not both 2, and $q \geq 3$ satisfying

$$\frac{1}{p_0} + \frac{1}{p_1} + \frac{2}{q} > 1,$$

  where $p_0 = p_1$ if q is odd. Using Shephard and Todd's numbering, these groups are $G_4, G_5, G_6, G_8, G_9, G_{10}, G_{14}, G_{16}, G_{17}, G_{18}, G_{20}, G_{21}$.
- $G(r, 1, n) = \mathbb{Z}/r\mathbb{Z} \wr \mathfrak{S}_n$ with $r > 2$. The group can be represented as $n \times n$ monomial matrices with nonzero entries rth roots of unity, and its symbol is $r[4]2[3]2 \ldots 2[3]2$.
- $2[4]3[3]3 = G_{26}$
- $3[3]3[3]3 = G_{25}$
- $3[3]3[3]3[3]3 = G_{32}$.

The following list summarizes notation and assumptions that will remain fixed when dealing with Shephard groups.

- W is a Shephard group.
- $\mathscr{P}$ is a non-starry regular complex polytope with symmetry group W.
- $K(\mathscr{P})$ is the flag complex of $\mathscr{P}$, consisting of all flags of (proper) faces.
- $\mathcal{B} = B_0 \subset B_1 \subset \ldots \subset B_{\ell-1}$ is a chosen base flag in $K(\mathscr{P})$.
- R is a set of distinguished generators for W corresponding to $\mathcal{B}$. One can choose R to satisfy the presentation found in the classification above, but doing so is unnecessary.
- Let $U \subseteq R$ with $U = \{r_{i_1}, r_{i_2}, \ldots, r_{i_k}\}$ and $i_1 < i_2 < \ldots < i_k$. Then

$$\mathcal{B}_U \overset{\text{def}}{=} B_{i_1} \subset B_{i_2} \subset \ldots \subset B_{i_k}.$$

- $\mathscr{L}_W$ is the lattice of hyperplane intersections for W under reverse inclusion.

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5.2. Shephard systems. In this section we give the analogue of Theorem 4.1 for Shephard groups (Theorem 5.3).

If W is the symmetry group of a *real* regular polytope $\mathscr{P}$, then the Coxeter complex Σ is obtained by cutting the real sphere $\mathbb{S}^{\ell-1}$ by the reflecting hyperplanes, and a radial projection sends Σ homeomorphically onto the barycentric subdivision of $\mathscr{P}$. Moreover, it is a geometric realization of the poset of *standard cosets*

$$\Delta(W, R) = \{gW_J\}_{J \subseteq R}$$

ordered by reverse inclusion. This section presents the analogous picture for Shephard groups, as established by Orlik [19] and Orlik–Reiner–Shepler [22].

The *vertices* of a face F of $\mathscr{P}$ are the vertices of $\mathscr{P}$ lying on F, and the *centroid* O_F of F is the average of its vertices. Centroids play an important role in what follows.

Definition 5.1. A *Shephard system* is a triple (W, R, Λ) with the following properties:
  (i) W is the symmetry group of a nonstarry regular complex polytope $\mathscr{P}$.
  (ii) R is a set of distinguished generators corresponding to a chosen base flag

$$\mathcal{B} = B_0 \subset B_1 \subset \ldots \subset B_{\ell-1}.$$

(iii) $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_{\ell-1}\}$ is defined by setting $\lambda_i = O_{B_i}$.

Such triples are framed systems:

Proposition 5.2. *A Shephard system is a framed system.*

Proof. Recall from Section 5.1 that for $0 \leq k \leq \ell - 1$, the reflection $r_k \in R$ stabilizes

$$B_0 \subset \ldots \subset \hat{B}_k \subset \ldots \subset B_{\ell-1}.$$

In particular, the centroid O_i of B_i is fixed by all r_k with $k \neq i$. In other words,

$$O_i \in H_0 \cap \ldots \cap \hat{H}_i \cap \ldots \cap H_{\ell-1}. \qquad \square$$

Let $B(\mathscr{P})$ denote the (topological) subspace of V that consists of all real hulls of centroids of flags under inclusion, i.e.,

$$B(\mathscr{P}) = \bigcup_{(F_{i_1} \subset \ldots \subset F_{i_k}) \in K(\mathscr{P})} \mathrm{Hull}(O_{F_{i_1}}, \ldots, O_{F_{i_k}}).$$

We now state the analogue of Theorem 4.1 for Shephard groups:

Theorem 5.3. *Let (W, R, Λ) be a Shephard system. Then the following hold:*
  (i) (W, R, Λ) is strongly stratified.
  (ii) $\rho(\Delta) = B(\mathscr{P})$.
  (iii) $\Delta := \Delta(W, R)$ is homotopy Cohen–Macaulay.
  (iv) (W, R, Λ) is locally conical.

In particular, Theorem 3.7, Theorem 3.8, and Theorem 3.11 apply to all Shephard systems.

Note that Figure 11 shows why the non-starry assumption is necessary; indeed, ρ fails to be an embedding in the starry case when $\Lambda = \{O_{B_0}, O_{B_1}\}$ and $B_0 \subset B_1$ is a maximal flag.

The remainder of this section explains (i)–(iii). Property (iv) will be established in the next subsection. During our discussion, the reader should take note of our use of Theorems 5.4 and 5.7, two *uniformly stated* theorems that are *proven case by case*. In particular,

Theorem 5.4 relies on a theorem of Orlik and Solomon which says that a Shephard group and its *associated Coxeter group* have the same discriminant matrices, a result which relies on the classification of Shephard groups; see [21]. However, up to the use of Theorems 5.4 and 5.7, our approach for Theorem 5.3 is case-free.

For each Shephard group, the invariant f_1 of smallest degree d_1 is unique, up to constant scaling. For example, if W has a real form, then $f_1 = x_1^2 + \ldots + x_\ell^2$ for some suitable set of coordinates. The *Milnor fiber of W* is defined to be $f_1^{-1}(1)$, where f_1 is regarded as a map $f_1 : V \to \mathbb{C}$. In [19], Orlik constructs a W-equivariant strong deformation retraction of the Milnor fiber $f_1^{-1}(1)$ onto a simplicial complex Γ homeomorphic to $B(\mathscr{P})$, which he shows is a geometric realization of the flag complex $K(\mathscr{P})$.

Theorem 5.4 (Orlik [19])**.** *Let $W \subset \mathrm{GL}(V)$ be a Shephard group with invariant $f_1 : V \to \mathbb{C}$ of smallest degree. Then there exists a simplicial complex $\Gamma \subset f_1^{-1}(1)$ called the* Milnor fiber complex *that contains the vertices of $\mathscr{P}$ and has the following properties:*

   (1) (a) There is an equivariant strong deformation retract $\pi : f_1^{-1}(1) \to \Gamma$.
           (b) For each $X \in \mathscr{L}_W$, the set $\Gamma_X := \Gamma \cap X$ is a subcomplex of Γ, and π restricts to a strong deformation retract of $f_1^{-1}(1) \cap X$ onto Γ_X.
   (2) Let Γ^k and Γ_X^k denote the k-skeleton of each complex. Then
           (a) $\Gamma_X^k = \Gamma^k \cap X$ for all k, and
           (b) $\Gamma^k \setminus \Gamma^{k-1} = \bigcup_{\dim X = k+1} (\Gamma_X^k \setminus \Gamma_X^{k-1})$ is a disjoint union.
   (3) (a) Γ is W-equivariantly homeomorphic to $B(\mathscr{P})$, and
           (b) $B(\mathscr{P})$ is a geometric realization of the flag complex $K(\mathscr{P})$ via

$$F_{i_1} \subset F_{i_2} \subset \ldots \subset F_{i_k} \longmapsto \mathrm{Hull}(O_{F_{i_1}}, O_{F_{i_2}}, \ldots, O_{F_{i_k}}).$$

Parts (1) and (2) are [19, Thm. 4.1(i)–(ii)], while (3) uses the proof of [19, Thm. 5.1]. A nice discussion of the related theory is found in [22].

Using the additional property that $f_1^{-1}(1)$ has an isolated critical point at the origin, Orlik was able to describe the topology of the flag complex $K(\mathscr{P})$ as a pure bouquet of spheres. Each open interval in $\mathscr{P}$ is another poset of faces of a regular complex polytope [14, p. 116] and so (by [10, Cor. 3.5], e.g.,) it follows that $K(\mathscr{P})$ is in fact homotopy Cohen–Macaulay:

Theorem 5.5 (Orlik [19])**.** *$K(\mathscr{P})$ is homotopy Cohen–Macaulay, and is homotopy equivalent to a wedge of $(d_1 - 1)^\ell$ spheres of dimension $\ell - 1$.*

We establish (i) and (iii) of Theorem 5.3 by combining Theorem 5.4(3) and Theorem 5.5 with the following result.

Theorem 5.6 (Orlik–Reiner–Shepler [22])**.** *Let W be a Shephard group of $\mathscr{P}$, and let $\mathcal{B}$ be a base flag with corresponding distinguished generating set R. Then the map*

$$\phi : K(\mathscr{P}) \longrightarrow \{gW_J \ : \ g \in W, \ J \subseteq R\}$$

$$g(B_{i_1} \subset \ldots \subset B_{i_s}) \overset{\phi}{\longmapsto} gW_{R \setminus \{r_{i_1}, \ldots, r_{i_s}\}}$$

is a W-equivariant isomorphism.

The crux of the proof is that

$$\mathrm{Stab}_W(B_{j_1} \subset B_{j_2} \subset \ldots \subset B_{j_s}) = W_{R \setminus \{r_{j_1}, r_{j_2}, \ldots, r_{j_s}\}}.$$

The type function on $K(\mathscr{P})$ is naturally given by

$$\text{type}(B_{i_1} \subset B_{i_2} \subset \ldots \subset B_{i_s}) = \{r_{i_1}, r_{i_2}, \ldots, r_{i_s}\}.$$

Proof of Theorem 5.3 (ii), (iii), *and the well-framed component of* (i). Notice that ρ factors as

$$\Delta \xrightarrow{\phi^{-1}} K(\mathscr{P}) \xrightarrow{\sim} B(\mathscr{P}) \hookrightarrow V,$$

where ϕ is as in Theorem 5.6, and $K(\mathscr{P}) \xrightarrow{\sim} B(\mathscr{P})$ is provided by Theorem 5.4(3)(a). Hence the triple (W, R, Λ) is well-framed and $\rho(\Delta) = B(\mathscr{P})$. Employing Theorem 5.5 gives (iii). $\square$

All that remains for establishing (i)–(iii) is to show that each $X \in \mathscr{L}_W$ contains the image under ρ of a $(\dim X - 1)$-simplex of Δ. This follows from the following beautiful theorem that merges work of Orlik–Solomon [20, Thm. 6] and Orlik [19, Thm. 4.1(iii)]; this amalgamation appears in the latter paper of Orlik.

Theorem 5.7 (Orlik–Solomon). *Let W the symmetry group of a nonstarry regular complex polytope $\mathscr{P}$. Let $X \in \mathscr{L}_W$, and write $\dim X = n$. Then there exists strictly positive integers $b_1^X, b_2^X, \ldots, b_n^X$ such that*

$$(8) \qquad |\Gamma_X^{n-1} \setminus \Gamma_X^{n-2}| = (m_1 + b_1^X)(m_1 + b_2^X) \ldots (m_1 + b_n^X),$$

where $m_1 = d_1 - 1$ and Γ_X^k is the k-skeleton of the restricted Milnor fiber complex $X \cap \Gamma$.

Proof of Theorem 5.3(i). Consider $X \in \mathscr{L}_W$ with $\dim X = n \geq 1$. By Theorem 5.7, $\Gamma_X^{n-1} \setminus \Gamma_X^{n-2}$ is nonempty if $m_1 \geq 0$, as this implies that the right side of (8) is strictly positive. But this is clear, since $d_1 = \deg f_1 \geq 1$ for any set of basic invariants $f_1, f_2, \ldots, f_\ell$. (In fact, $d_1 \geq 2$, with equality if and only if W is a real reflection group.)

By Theorem 5.4, each $(n-1)$-simplex of $\Gamma \cap X$ corresponds to an $(n-1)$-simplex of $B(\mathscr{P}) \cap X$. Hence, $(B(\mathscr{P}) \cap X)^{n-1}$ is nonempty. Since $\rho(\Delta) = B(\mathscr{P})$, it follows by considering dimension that $X = \text{Supp}(\sigma)$ for some $\sigma \in \Delta$. $\square$

5.3. **Shephard systems are locally conical.** This section is dedicated to proving (iv) of Theorem 5.3. Throughout, (W, R, Λ) will be a fixed Shephard system, and F_i will denote an i-dimensional face of $\mathscr{P}$. Because we will need to work with faces of $\mathscr{P}$ instead of centroids, we start with some straightforward results relating the two. The most important of these results says that centroids of a maximal flag (together with the origin O_{F_ℓ}) form an *orthoscheme*; see [14, p. 116]. More precisely, we have the following:

Proposition 5.8 (Coxeter). *Let $\mathscr{P}$ be a regular complex polytope, and let*

$$\mathcal{F} = F_0 \subset F_1 \subset \ldots \subset F_{\ell-1}$$

be a maximal flag of faces. Then the vectors

$$(9) \qquad O_{F_{\ell-1}} - O_{F_\ell}, \qquad O_{F_{\ell-2}} - O_{F_{\ell-1}}, \qquad \ldots, \qquad O_{F_0} - O_{F_1}$$

form an orthogonal basis for V.

We get the next lemma by taking partial sums in (9).

Lemma 5.9. *Let $F_0 \subset F_1 \subset \ldots \subset F_{\ell-1}$ be a maximal flag of a regular polytope $\mathscr{P}$. Then the centroids $O_{F_0}, O_{F_1}, \ldots, O_{F_{\ell-1}}$ form a basis for $V = \mathbb{C}^\ell$.*

Two particularly important results follow from Lemma 5.9.

Corollary 5.10. $F_i = \operatorname{AffSpan}(O_{F_0}, \ldots, O_{F_i})$ *for all* $i \geq 0$.

Corollary 5.11. *If $\mathcal{F}_1, \mathcal{F}_2$ are two subflags of a fixed flag $\mathcal{F}$, then*

$$\operatorname{Span}\left(\{O_F\}_{F \in \mathcal{F}_1}\right) \cap \operatorname{Span}\left(\{O_F\}_{F \in \mathcal{F}_2}\right) = \operatorname{Span}\left(\{O_F\}_{F \in \mathcal{F}_1 \cap \mathcal{F}_2}\right)$$

and

$$\operatorname{AffSpan}\left(\{O_F\}_{F \in \mathcal{F}_1}\right) \cap \operatorname{AffSpan}\left(\{O_F\}_{F \in \mathcal{F}_2}\right) = \operatorname{AffSpan}\left(\{O_F\}_{F \in \mathcal{F}_1 \cap \mathcal{F}_2}\right).$$

We are now ready for the main tool that will be used in the proof of Theorem 5.3(iv):

Proposition 5.12. *Let $X \in \mathscr{L}_W$ of dimension $k > 0$, and suppose that $\mathcal{F} = F_{i_1} \subset \ldots \subset F_{i_k}$ and $\mathcal{F}' = F'_{j_1} \subset \ldots \subset F'_{j_k}$ are two k-flags with*

$$X = \operatorname{Span}(O_{F_{i_1}}, \ldots, O_{F_{i_k}}) = \operatorname{Span}(O_{F'_{j_1}}, \ldots, O_{F'_{j_k}}).$$

Set $i_{k+1} = \ell$. For $1 \leq s \leq k$, if F is a face such that $F_{i_s} \subseteq F \subsetneq F_{i_{s+1}}$, then one of the following holds:

(a) $F'_{j_s} = F_{i_s}$.
(b) $F'_{j_t} \not\subseteq F$ for all t.

Proof. Suppose that $F'_{j_t}, F_{i_s} \subseteq F \subsetneq F_{i_{s+1}}$ with s and t maximal. Using the definition of a polytope, we can extend $F_{i_1} \subset \ldots \subset F_{i_s} \subseteq F \subseteq \ldots \subseteq F_{i_k}$ to a maximal flag $\mathcal{F}$. By doing so, Corollaries 5.10 and 5.11 imply that

(10) $$F \cap X = \operatorname{AffSpan}(O_{F_{i_1}}, \ldots, O_{F_{i_s}}).$$

Similarly,

(11) $$F \cap X = \operatorname{AffSpan}(O_{F'_{j_1}}, \ldots, O_{F'_{j_t}}).$$

By comparing dimension, we have that $t = s$.

Our first claim is that F_{i_s}, F'_{j_s} are minimal faces of F containing $F \cap X$. Certainly the intersection is contained in F_{i_s} (and F'_{j_s}). Moreover, it contains the centroid $O_{F_{i_s}}$ (resp. $O_{F'_{j_s}}$), which cannot be contained in any proper face of F_{i_s} (resp. F'_{j_s}) by Corollary 5.10.

Assume without loss of generality that $i_s \leq j_s$. From the equalities of (10) and (11), we have $O_{F'_{j_s}} \in F_{i_s}$. The definition of a polytope implies the existence of a face $\widetilde{F}_{j_s} \supseteq F_{i_s}$, which must therefore necessarily contain both $O_{\widetilde{F}_{j_s}}$ and $O_{F'_{j_s}}$. We claim that $F'_{j_s} = \widetilde{F}_{j_s}$. This follows immediately if $O_{F'_{j_s}} = O_{\widetilde{F}_{j_s}}$, since faces and centroids determine each other; see Theorem 5.4(3)(b), for example. The other case is slightly more work.

Suppose that $O_{F'_{j_s}} \neq O_{\widetilde{F}_{j_s}}$. Extend $\widetilde{F}_{j_s}$ to a maximal flag $\widetilde{\mathcal{F}}$ and recall that the centroids for $\widetilde{\mathcal{F}}$ form an orthoscheme. Combining Corollary 5.10 with Proposition 5.8, it follows that the vector of $O_{\widetilde{F}_{j_s}}$ is perpendicular to $\widetilde{F}_{j_s}$. As $O_{F'_{j_s}}$ is also in $\widetilde{F}_{j_s}$, this implies that $O_{F'_{j_s}} - O_{\widetilde{F}_{j_s}}$ is perpendicular to $O_{\widetilde{F}_{j_s}}$. Hence,

$$\begin{aligned}
|O_{F'_{j_s}}|^2 &= |O_{F'_{j_s}} - O_{\widetilde{F}_{j_s}} + O_{\widetilde{F}_{j_s}}|^2 \\
&= \langle O_{F'_{j_s}} - O_{\widetilde{F}_{j_s}} + O_{\widetilde{F}_{j_s}}, O_{F'_{j_s}} - O_{\widetilde{F}_{j_s}} + O_{\widetilde{F}_{j_s}} \rangle \\
&= |O_{F'_{j_s}} - O_{\widetilde{F}_{j_s}}|^2 + |O_{\widetilde{F}_{j_s}}|^2.
\end{aligned}$$

Because $O_{F'_{j_s}} \neq O_{\widetilde{F}_{j_s}}$, we therefore have that

$$|O_{F'_{j_s}}| > |O_{\widetilde{F}_{j_s}}|.$$

This contradicts regularity, which tells us that there is a unitary $g \in W$ with $g\widetilde{F}_{j_s} = F'_{j_s}$, i.e., with $gO_{\widetilde{F}_{j_s}} = O_{F'_{j_s}}$.

Having established that $F'_{j_s} = \widetilde{F}_{j_s} \supseteq F_{i_s}$, the minimality of F'_{j_s} forces equality. $\qquad \square$

Proof of Theorem 5.3(iv). Let $U \subseteq R$ be nonempty. Identifying Δ with $K(\mathscr{P})$, we have $W_{R \setminus U}$ corresponds to $\mathcal{B}_U$, and

$$\Delta^U = \mathrm{St}_\Delta(W_{R \setminus U}) = \mathrm{St}_{K(\mathscr{P})}(\mathcal{B}_U),$$

with $\mathrm{Supp} : \mathrm{Face}(\Delta^U) \longrightarrow \Pi^U \setminus \{\hat{1}\}$ given by $F_{i_1} \subset \ldots \subset F_{i_s} \longmapsto \mathrm{Span}(O_{F_{i_1}}, \ldots, O_{F_{i_s}})$.

Let $X \in \Pi^U \setminus \{\hat{1}\}$. We claim that for some face F_t of $\mathscr{P}$, the Quillen fiber

$$\Phi = X \cap \Delta^U$$

has F_t as a cone point. This implies the desired claim that (W, R, Λ) is locally conical.

From the definition of Π^U and Theorem 5.4(3), we can write

$$X = \mathrm{Span}(O_{F_{t_1}}, \ldots, O_{F_{t_k}})$$

for some k-flag $\mathcal{F} = F_{t_1} \subset \ldots \subset F_{t_k}$ in Δ^U. Because $\mathcal{F}$ can be extended to a flag in Δ containing $\mathcal{B}_U$, we can fix a nontrivial face $F \in \mathcal{B}_U$ and choose m such that one of the following holds:

(a) F_{t_m} is the maximal face of $\mathcal{F}$ that is weakly contained in F, or
(b) F_{t_m} is the minimal face of $\mathcal{F}$ that weakly contains F.

We claim that F_{t_m} is a cone point of Φ. To establish this, suppose $\mathcal{F}'$ is another k-flag in Δ^U with support X. Using Lemma 5.12, we will show that $F_{t_m} \in \mathcal{F}'$.

Suppose first that we are in case (a), i.e., $F_{t_m} \subseteq F$. Because $\mathcal{F}'$ can be extended to a maximal flag containing $\mathcal{B}_U$, either

(1) an element of $\mathcal{F}'$ is weakly contained in F, or
(2) each element of $\mathcal{F}'$ strictly contains F.

However, (2) is not a valid possibility. Supposing otherwise, we can extend $\mathcal{F}'$ by F_{t_m} to get a strictly larger flag with support equal to X. Considering dimension yields a contradiction. In the case of (1), Lemma 5.12 shows that $F_{t_m} \in \mathcal{F}'$.

If we are instead in case (b), i.e., $F_{t_m} \supseteq F$, we can use the same argument, applied to the dual polytope $\mathscr{P}^*$, to conclude that $F_{t_m} \in \mathcal{F}'$. $\qquad \square$

6. Ribbon representations revisited

This section relates ribbon representations of Coxeter and Shephard groups to the group algebra, the exterior powers of the reflection representation, and the coinvariant algebra.

Recall that the group algebra $\mathbb{C}[G]$ of a finite group G over $\mathbb{C}$ is semisimple, and thus the Grothendieck group $R(G)$ of the category of finite dimensional G-representations is the free $\mathbb{Z}$-module with basis the isomorphism classes of irreducible G-modules. The map sending each irreducible module to its character linearly extends to an isomorphism of $\mathbb{C} \otimes_\mathbb{Z} R(G)$ onto the space of complex class functions on G. In fact, this is an isomorphism in the category of finite-dimensional Hilbert spaces, where the form $\langle -, - \rangle$ on $\mathbb{C} \otimes_\mathbb{Z} R(G)$ is that

for which the irreducible G-modules form an orthonormal basis, and the form $\langle -, - \rangle$ on class functions is given by $\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}$. In what follows, we make no notational distinction between elements of $\mathbb{C} \otimes_{\mathbb{Z}} R(G)$ and class functions.

6.1. **Solomon's group algebra decomposition.** Let (W, R) be a simple system, and ϵ be the sign representation, defined by $\epsilon(r) = -1$ for $r \in R$. In [26], Solomon showed that

$$(12) \qquad \mathbb{C}W = \bigoplus_{T \subseteq R} \mathbb{C}W c_T \quad \text{and} \quad \mathbb{C}W c_T \cong_W \sum_{T \subseteq J \subseteq R} (-1)^{|J \setminus T|} \mathrm{Ind}_{W_J}^W \mathbb{1}$$

for $c_T := a_T b_{R \setminus T}$, where

$$a_T = \frac{1}{|W_T|} \sum_{g \in W_T} g \quad \text{and} \quad b_{R \setminus T} = \frac{1}{|W_{R \setminus T}|} \sum_{g \in W_{R \setminus T}} \epsilon(g) g.$$

The W-modules $\mathbb{C}W c_T$ are known as *ribbon representations* due to their alternate description when $W = \mathfrak{S}_n$ and $r_i = (i, i+1)$ for $1 \leq i \leq n-1$. Considering this case, let $T \subseteq R$, write $R \setminus T = \{r_{i_1}, r_{i_2}, \ldots, r_{i_j}\}$ with $i_1 < i_2 < \ldots < i_j$, and let λ/μ be the ribbon skew shape corresponding to composition $(i_1, i_2 - i_1, \ldots, i_j - i_{j-1}, n - i_j)$; see Figure 12. Filling λ/μ with $1, 2, \ldots, n$ in increasing order from southwest to northeast gives a tableau whose rows are stabilized by W_T and whose columns are stabilized by $W_{R \setminus T}$, so that c_T is the *Young symmetrizer* $c_{\lambda/\mu}$ and $\mathbb{C}W c_T$ is the $\mathfrak{S}_n$-Specht module of ribbon skew shape λ/μ.

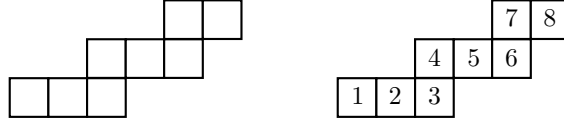

FIGURE 12. The ribbon skew shape and filling for composition $(3, 3, 2)$.

For a simple system or Shephard system (W, R), define $\chi^T = \widetilde{H}_{|T|-1}(\Delta_T)$ for $T \subseteq R$. We call these *ribbon representations*, noting that in the case of Coxeter groups

$$\mathbb{C}W c_T \cong_W \sum_{T \subseteq J \subseteq R} (-1)^{|J \setminus T|} \mathrm{Ind}_{W_J}^W \mathbb{1} = \sum_{J \subseteq R \setminus T} (-1)^{|(R \setminus T) \setminus J|} \mathrm{Ind}_{W_{R \setminus J}}^W \mathbb{1} \cong_W \chi^{R \setminus T}.$$

Regarding Solomon's group algebra decomposition, applying Möbius inversion to the equality

$$\chi^R = \sum_{T \subseteq R} (-1)^{|R \setminus T|} \mathrm{Ind}_{W_{R \setminus T}}^W \mathbb{1}$$

gives the following theorem.

Theorem 6.1. *For (W, R) a simple system or Shephard system, $\mathbb{C}W \cong_W \bigoplus_{T \subseteq R} \chi^T$.* $\qquad \square$

6.2. **Exterior powers of the reflection representation.** Another extension of a main theorem in [26] is the following.

Theorem 6.2. *Let (W, R) be a simple system or Shephard system. For $T \subseteq R$, the ribbon representation χ^T contains a unique irreducible submodule isomorphic to $\bigwedge^{|T|} V$ and has no submodule isomorphic to $\bigwedge^p V$ for $p \neq |T|$.*

One can follow the same proof as in [26], replacing (12) with Theorem 6.1, but here we give a shorter and simpler proof.

Proof of Theorem 6.2. Since $\bigwedge^{|T|}V$ occurs in $\mathbb{C}W$ with multiplicity equal to its dimension $\binom{|R|}{|T|}$, by Theorem 6.1 it will suffice to show $\langle \chi^T, \bigwedge^{|T|}V \rangle = 1$ for each $T \subseteq R$. Fixing $T \subseteq R$, so that $\chi^T = \sum_{J \subseteq T}(-1)^{|T \setminus J|}\mathrm{Ind}_{W_{R \setminus J}}^W \mathbb{1}$, it suffices to show for each subset $J \subseteq T$ that

$$(13) \qquad \left\langle \mathrm{Ind}_{W_{R \setminus J}}^W \mathbb{1} \,,\, \bigwedge\nolimits^{|T|}V \right\rangle = \begin{cases} 1 & \text{if } J = T, \\ 0 & \text{otherwise.} \end{cases}$$

If $J = R$, then $\mathrm{Ind}_{W_{R \setminus J}}^W \mathbb{1}$ is the regular representation and $\dim \bigwedge^{|T|}V = 1$, so (13) follows.

Assuming that $J \neq R$, there are disjoint nonempty sets $I_j \subseteq R \setminus J$ such that

$$W_{R \setminus J} \cong W_{I_1} \times \ldots \times W_{I_n}$$

with each W_{I_j} acting irreducibly on $V_j := (\bigcap_{s \in I_j} H_s)^{\perp}$, which yields a decomposition

$$V = V^{W_{R \setminus J}} \oplus V_1 \oplus \ldots \oplus V_n$$

on which each W_{I_j} acts trivially on all factors except V_j; see [18, Ch. 1]. The resulting decomposition of the exterior power

$$\bigwedge\nolimits^{|T|}V \cong \bigoplus_{i_1 + \ldots + i_n + m = |T|} \bigwedge\nolimits^m(V^{W_{R \setminus J}}) \otimes \bigwedge\nolimits^{i_1}V_1 \otimes \ldots \otimes \bigwedge\nolimits^{i_n}V_n,$$

combined with Frobenius reciprocity, implies that

$$\left\langle \mathrm{Ind}_{W_{R \setminus J}}^W \mathbb{1} \,,\, \bigwedge\nolimits^{|T|}V \right\rangle = \sum_{m=0}^{|J|} \binom{|J|}{m} \sum_{i_1 + \ldots + i_n = |T| - m} \prod_{j=1}^n \left\langle \mathbb{1} \,,\, \bigwedge\nolimits^{i_j}V_j \right\rangle_{W_{I_j}}.$$

By a theorem of Steinberg [11, Ch. 5, §2, Exercise 3], the W_{I_j}-modules $\bigwedge^k V_j$ are irreducible and distinct for $0 \leq k \leq \dim V_j$. Hence $\langle \mathbb{1} \,,\, \bigwedge^{i_j}V_j \rangle_{W_{I_j}} = \delta_{i_j,0}$ and (13) follows. $\qquad\square$

6.3. **Expressing the ribbon decomposition of the coinvariant algebra.** A central object in invariant theory is the coinvariant algebra S/S_+^W of a finite subgroup $W \subset \mathrm{GL}(V)$, which is the graded quotient of S by the ideal S_+^W generated by homogeneous invariants of positive degree. Recall that W is a reflection group if and only if S/S_+^W affords the regular representation as an ungraded module.

This section concerns the decomposition of the coinvariant algebra of a Shephard group W into ribbon representations. More precisely, we give a determinantal expression for the multivariate generating function

$$W(\mathbf{t}, q) \overset{\mathbf{def}}{=} \sum_{T \subseteq R} \langle \chi^T, S/S_+^W \rangle(q) \, \mathbf{t}^T,$$

where $\mathbf{t}^T$ is defined below, and $\langle \chi, M \rangle(q)$ denotes the graded inner product $\sum_{d \geq 0} \langle \chi, M_d \rangle q^d$ of an element $\chi \in \mathbb{C} \otimes_{\mathbb{Z}} R(W)$ and a graded W-module $M = \bigoplus_{d=0}^{\infty} M_d$.

For a Shephard system (W, R) and subset $J \subseteq R$, define

$$\mathbf{t}^J = \prod_{r_i \in J} t_i, \quad (\mathbf{1} - \mathbf{t})^J = \prod_{r_i \in J}(1 - t_i), \quad W_J(q) = \mathrm{Hilb}(S/S_+^{W_J}, q), \quad W_{[i,j]} = W_{\{r_k \,:\, i \leq k \leq j\}},$$

where $W_\varnothing$ is the trivial subgroup, we set $W(q) = W_R(q)$, and the *Hilbert series* of a graded module $M = \bigoplus_{d=0}^{\infty} M_d$ is the formal power series $\mathrm{Hilb}(M, q) := \sum_{d \geq 0} M_d q^d$.

Recall that the generators of a Shephard system (W, R) inherit an indexing from the associated chosen flag of faces $B_0 \subset B_1 \subset \ldots \subset B_{\ell-1}$. However, in what follows it will be convenient to shift all indices by 1, thus writing $R = \{r_1, r_2, \ldots, r_\ell\}$.

Theorem 6.3. *Let (W, R) be a Shephard system. Then*

$$(14) \qquad W(\mathbf{t}, q) = W(q) \cdot \det \begin{bmatrix} 1 & \frac{1}{W_{[1,1]}(q)} & \frac{1}{W_{[1,2]}(q)} & \cdots & \frac{1}{W_{[1,\ell]}(q)} \\ t_1 - 1 & t_1 & \frac{t_1}{W_{[2,2]}(q)} & \cdots & \frac{t_1}{W_{[2,\ell]}(q)} \\ 0 & t_2 - 1 & t_2 & \cdots & \frac{t_2}{W_{[3,\ell]}(q)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_\ell - 1 & t_\ell \end{bmatrix}.$$

Remark 6.4. For a (finite) Coxeter system (W, R), the *multivariate q-Eulerian distribution* is defined to be

$$\mathrm{Eul}(\mathbf{t}, q) = \sum_{g \in W} \mathbf{t}^{\mathrm{Des}(g)} q^{\ell(g)},$$

where $\ell(g) = \min\{m : r_{i_1} \ldots r_{i_m} = g\}$ is the usual length function on Coxeter groups, and $\mathrm{Des}(g) = \{r \in R : \ell(gr) < \ell(g)\}$ is the descent set of g. In the case of real Shephard groups, Reiner [23], following Stembridge [30], established Theorem 6.3 with $\mathrm{Eul}(\mathbf{t}, q)$ in place of $W(\mathbf{t}, q)$. Thus,

$$(15) \qquad\qquad\qquad \mathrm{Eul}(\mathbf{t}, q) = W(\mathbf{t}, q)$$

for real Shephard groups. Extending (15) to other Shephard groups is a problem of considerable interest.

Proof of Theorem 6.3. Fix $T \subseteq R$ and consider the coefficient of $\mathbf{t}^T$ in $W(\mathbf{t}, q)$. By Frobenius reciprocity, we have

$$\langle \chi^T, S/S_+^W \rangle(q) = \sum_{J \subseteq T} (-1)^{|T \setminus J|} \langle \mathrm{Ind}_{W_{R \setminus J}}^W \mathbb{1}, S/S_+^W \rangle(q)$$

$$= \sum_{J \subseteq T} (-1)^{|T \setminus J|} \langle \mathbb{1}, S/S_+^W \rangle_{W_{R \setminus J}}(q).$$

Recall that for a reflection group $G \subset \mathrm{GL}(V)$, one has $S \cong S^G \otimes_{\mathbb{C}} S/S_+^G$ as graded G-modules. It follows that

$$W_J(q) = \frac{\mathrm{Hilb}(S, q)}{\mathrm{Hilb}(S^{W_J}, q)}$$

for any $J \subseteq R$, and that the graded W-module S/S_+^W has graded character

$$\chi(g) = \sum_{d \geq 0} \mathrm{Tr}(g|_{(S/S_+^W)_d}) q^d = \frac{1}{\det(1 - gq)} \frac{1}{\mathrm{Hilb}(S^W, q)}.$$

Therefore,

$$\langle \mathbb{1}, S/S_+^W \rangle_{W_{R \setminus J}}(q) = \frac{1}{\mathrm{Hilb}(S^W, q)} \langle \mathbb{1}, S \rangle_{W_{R \setminus J}}(q) = \frac{\mathrm{Hilb}(S^{W_{R \setminus J}}, q)}{\mathrm{Hilb}(S^W, q)} = \frac{W(q)}{W_{R \setminus J}(q)}.$$

Consider now the right-hand side of (14). From the usual permutation expansion of the determinant, we have the following general equation:

$$
\det \begin{bmatrix}
a_{01} & a_{02} & a_{03} & \cdots & a_{0,n+1} \\
a_{11} & a_{12} & a_{13} & \cdots & a_{1,n+1} \\
0 & a_{22} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & a_{nn} & a_{n,n+1}
\end{bmatrix}
= \sum_{1 \le i_1 < \ldots < i_j \le n} (-1)^{n-j} a_{0,i_1} a_{i_1,i_2} \ldots a_{i_j,n+1} \prod_{\substack{1 \le i \le n \\ i \ne i_1,\ldots,i_j}} a_{ii}.
$$

The right-hand side of (14) is therefore equal to

$$
W(q) \sum_{1 \le i_1 < \ldots < i_j \le \ell} (-1)^{\ell-j} \frac{t_{i_1} \ldots t_{i_j}}{W_{[1,i_1-1]}(q) W_{[i_1+1,i_2-1]}(q) \ldots W_{[i_j+1,\ell]}(q)} \prod_{\substack{1 \le i \le \ell \\ i \ne i_1,\ldots,i_j}} (t_i - 1),
$$

which, using the fact that $r_i r_j = r_j r_i$ for $|i - j| \ge 2$, can be written as

$$
W(q) \sum_{J \subseteq R} (-1)^{|R \setminus J|} \frac{\mathbf{t}^J}{W_{R \setminus J}(q)} (\mathbf{t} - \mathbf{1})^{R \setminus J}.
$$

Taking the coefficient of $\mathbf{t}^T$ gives $\sum_{J \subseteq T} (-1)^{|T \setminus J|} \frac{W(q)}{W_{R \setminus J}(q)}$. $\qquad\square$

7. Remarks and questions

7.1. An interesting family of well-framed systems.
In Section 2.2 we introduced well-framed and strongly stratified systems (W, R, Λ). The well-framed systems subsequently studied were additionally strongly stratified, locally conical, and produced Cohen–Macaulay complexes $\Delta(W, R)$. However, there are well-framed systems (W, R) lacking many of these properties, including that of $\Delta(\Pi_T^U \setminus \{\hat{1}\})$ being homotopy equivalent to Δ_T^U. We discuss some of these here.

Consider $\mathscr{P}_n$ the boundary of the standard n-simplex having vertices labeled with the set $\{1, 2, \ldots, n+1\}$. Its symmetry group is $\mathfrak{S}_{n+1}$, and a minimal generating set of reflections corresponds to a spanning tree T of the complete graph K_{n+1} on $\{1, 2, \ldots, n+1\}$ by identifying an edge ij of T with the transposition (i, j); see [3]. We focus on the generating set

$$
R_n^\star := \{\underbrace{(1, n+1)}_{r_1}, \underbrace{(2, n+1)}_{r_2}, \ldots, \underbrace{(n, n+1)}_{r_n}\}
$$

of $\mathfrak{S}_{n+1}$ that corresponds to the star graph with center $n + 1$. We leave the proof of the following result to the reader.

Proposition 7.1. *Let $\mathscr{P}_n$ be as above. Let $v_1, v_2, \ldots, v_{n+1}$ be the vertices of $\mathscr{P}_n$, indexed so that v_i corresponds to the vertex labeled by i. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ be pairwise linearly independent over $\mathbb{R}$, and set $\Lambda := \{\alpha_1 v_1, \ldots, \alpha_n v_n\}$. Then $(\mathfrak{S}_{n+1}, R_n^\star, \Lambda)$ is well-framed.*

For $n \ge 3$, a system $(\mathfrak{S}_{n+1}, R_n^\star, \Lambda)$ as in Proposition 7.1 is neither locally conical nor strongly stratified. Moreover, for $n \ge 4$, it is no longer true that Δ_T^U is homotopy equivalent to $\Delta(\Pi_T^U \setminus \{\hat{1}\})$ for every $U, T \subseteq R$ with U nonempty. These assertions are straightforward after considering the following example.

Example 7.2. Let $(\mathfrak{S}_5, R_4^\star, \Lambda)$ be a system as in Proposition 7.1 with $n = 4$. The link of a 1-simplex in Δ has 6 vertices supporting 3 lines. The system is therefore not locally conical, and it even fails to be strongly stratified because the vertices of Δ support only the 5 complex lines spanned by the vertices of $\mathscr{P}_4$.

The link $\Delta_{\{r_1,r_2,r_3\}}^{\{r_4\}} = \mathrm{lk}_\Delta(W_{R\setminus\{r_4\}})$ of a vertex is isomorphic to $\Delta(\mathfrak{S}_4, R_3^\star)$. The latter is isomorphic to the 3×4 *chessboard complex*, which is known to be a 2-torus; see [3, Ex. 3.1] and [9, p. 30]. On the other hand, $\Pi_{\{r_1,r_2,r_3\}}^{\{r_4\}} \setminus \{\hat{1}\}$ is easily seen to be the face poset of the barycentric subdivision of $\mathscr{P}_3$, hence its order complex is a 2-sphere.

7.2. **The remaining reflection groups.** This subsection provides evidence that our main results partially extend to the remaining well-generated reflection groups.

We first show that $\Delta(W, R)$ is simplicial if each subgroup $W_J := \langle J \rangle$ with $J \subseteq R$ is *parabolic*, meaning the pointwise stabilizer of a subset of V. Note that saying W_J is parabolic is the same as saying that W_J is the stabilizer of $\cap_{r \in J} H_r$.

Proposition 7.3. *Let $W \subset \mathrm{GL}(V)$ be well-generated by R. If W_J is parabolic for all $J \subseteq R$, then $\Delta(W, R)$ is a simplicial complex.*

Proof. Suppose each W_J is parabolic. To establish the intersection property (6) from Section 3.3, first note that $\mathrm{Stab}(X_1) \cap \mathrm{Stab}(X_2) = \mathrm{Stab}(\mathrm{Span}(X_1, X_2))$ for $X_1, X_2 \subseteq V$. Thus,

$$
\begin{aligned}
\bigcap_{r \in R \setminus J} W_{R \setminus \{r\}} &= \bigcap_{r \in R \setminus J} \mathrm{Stab}\left(\bigcap_{s \in R \setminus \{r\}} H_s\right) \\
&= \mathrm{Stab}\left(\mathrm{Span}\left(\left\{\bigcap_{s \in R \setminus \{r\}} H_s \mid r \in R \setminus J\right\}\right)\right) \\
&= \mathrm{Stab}\left(\bigcap_{r \in J} H_r\right) \\
&= W_J,
\end{aligned}
$$

where the third equality follows from the obvious inclusion by comparing dimensions. $\square$

Broué–Malle–Rouquier [12, Appendix 2] gave *Coxeter-like presentations* for all (irreducible) complex reflection groups W. When W is a real reflection group, their presentation is the usual Coxeter presentation with R the set of reflections through walls of a Weyl chamber, and when W is a Shephard group, their presentation is the one given by the group's symbol $p_0[q_0]p_1 \ldots p_{\ell-2}[q_{\ell-2}]p_{\ell-1}$ from §5.1, so that (W, R) is a special choice of Shephard system. When W is well-generated, Broué–Malle–Rouquier's distinguished set of generators R has the property that each W_J ($J \subseteq R$) is parabolic [12, §1.7], making $\Delta(W, R)$ a simplicial complex by Proposition 7.3:

Theorem 7.4. *Let W be well-generated with distinguished set of generators R from [12, §1B]. Then $\Delta(W, R)$ is a simplicial complex.* $\square$

Remark 7.5. A well-generated group W and inclusion-minimal generating set R can give a simplicial complex even when W is not well-generated by R. For example, consider the rank-1 reflection group $\mathbb{Z}/6\mathbb{Z} = G(6, 1, 1)$ generated by a primitive 6th root of unity ζ. Then $R := \{\zeta^2, \zeta^3\}$ is inclusion-minimal with $|R| > \dim V$ and $\Delta(\mathbb{Z}/6\mathbb{Z}, R)$ simplicial.

Question 7.6. Let W be a well-generated group, and let R be a minimal generating set of reflections under inclusion. Is $\Delta(W, R)$ necessarily a simplicial complex?

Question 7.7. For which non-well-generated groups W is $\Delta(W, R)$ a simplicial complex for some R?

Let (W, R) be a well-generated system such that $\Delta(W, R)$ is a simplicial complex. Following (2), define

$$\text{Supp} : \Delta(W, R) \to \mathscr{L}_W \quad \text{by} \quad gW_J \mapsto V^{gW_Jg^{-1}}$$

and let

$$\Delta_T^U \overset{\text{def}}{=} \text{St}_{\Delta(W,R)}(W_{R \setminus U}) \mid_T \quad \text{and} \quad \Pi_T^U \overset{\text{def}}{=} \{V^{gW_Jg^{-1}} : gW_J \in \Delta_T^U\}$$

for $U, T \subseteq R$. Recall that we identify Δ_T^U with the poset of faces of a simplicial complex, and thus $\text{Face}(\Delta_T^U)$ is obtained from Δ_T^U by removing its unique bottom element W. Call (W, R) *(abstractly) locally conical* if for each $U, T \subseteq R$ with U nonempty, every Quillen fiber of $\text{Supp} : \text{Face}(\Delta_T^U) \to \Pi_T^U \setminus \{\hat{1}\}$ has a cone point. Note that if (W, R) is (abstractly) locally conical, then $\Delta(\Pi_T^U \setminus \{\hat{1}\})$ is $W_{R \setminus U}$-homotopy equivalent to Δ_T^U for all $U, T \subseteq R$ with U nonempty.

Conjecture 7.8. *For each well-generated reflection group W, there exists a well-generating R for which (W, R) is (abstractly) locally conical.*

Further, we predict the following partial extension of Theorems 4.1 and 5.3.

Conjecture 7.9. *For each well-generated reflection group W, there exists a generating set R and a frame Λ such that*

(i) $|R| = \dim V$.
(ii) (W, R, Λ) is strongly stratified.
(iii) (W, R, Λ) is locally conical.

7.3. **Shellability.** It is well known [5] that the Coxeter complex Δ for a finite Coxeter group is *shellable*, meaning that its facets can be ordered $F_1, F_2, \ldots, F_k$ so that the subcomplex $F_j \cap \left(\cup_{i=1}^{j-1} F_i \right)$ is pure of dimension $\dim \Delta - 1$ for all $j \geq 2$.

The question of whether the flag complex $K(\mathscr{P})$ of a regular complex polytope $\mathscr{P}$ is lexicographically shellable appears in [22, Question 16] and [6, p. 32]. By Section 5.2, $K(\mathscr{P})$ is isomorphic to $\Delta(W, R)$ for (W, R) a Shephard system for $\mathscr{P}$. It is straightforward to shell those of rank 2, as they are connected graphs, and it is also straightforward for $G(r, 1, n) = \mathbb{Z}/r\mathbb{Z} \wr \mathfrak{S}_n$. Those of Coxeter type are shellable, and for the other cases the author used a computer to find shellings:

Theorem 7.10. *Let (W, R) be a Coxeter or Shephard system. Then $\Delta(W, R)$ is shellable.*

Question 7.11. Is there a uniform way of shelling the flag complex $K(\mathscr{P})$ of a regular complex polytope? This would give a more direct proof that $K(\mathscr{P})$ is homotopy Cohen–Macaulay.

The following was inspired by a personal communication with T. Yun and [15, Section 8].

Question 7.12. Let (W, R) be a Coxeter or Shephard system. Is $\Pi_T^U \setminus \{\hat{1}\}$ shellable for all $U, T \subseteq R$?

Remark 7.13. [15] and its longer version [16] conclude with some interesting questions as well. In particular, the second paper ends by asking whether our techniques from this paper also work to extend some analogous results for knapsack partitions. We have not explored this here.

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E-mail address: `mill1966@math.umn.edu`

School of Mathematics, University of Minnesota, Minneapolis, MN 55455