Theorem 3 (Hopf's formula). *If a group G has a presentation $\langle S \mid R\rangle$, then*

$$H_2(G;\mathbb{Z}) \cong \frac{N\cap[F,F]}{[F,N]},$$

where $F = F(S)$ is the free group generated by S and $N = N(R)$ is the normal closure of R.

Applying Hopf's formula to Coxeter groups and Artin groups with their standard presentations, we may construct explicitly second homology classes as cosets $x[F,N]$ with $x \in N\cap[F,F]$ as above. We manage to find a set $\Omega(W)$ of generators of $H_2(W;\mathbb{Z})$ and a set $\Omega(A)$ of generators of $H_2(A;\mathbb{Z})$ such that the homomorphism $p_* : H_2(A;\mathbb{Z}) \to H_2(W;\mathbb{Z})$ induced by the natural map $p : A \to W$ maps $\Omega(A)$ onto $\Omega(W)$. Moreover we have by construction $\#\Omega(W) = n(\Gamma)$. On the other hand, Howlett proved the following.

Theorem 4 ([5]). *For an arbitrary Coxeter graph Γ, we have*

$$H_2(W(\Gamma);\mathbb{Z}) \cong \mathbb{Z}_2^{n(\Gamma)}.$$

Hence we know that $\Omega(W)$ is a basis of $H_2(W;\mathbb{Z})$, and we have proved that p_* is surjective. Theorem 2 follows without difficulties.

We expect that the above computation extends to higher homology of Artin groups. In fact, we have the similar ingredients: $H_3(W;\mathbb{Z})$ has been computed in [4] and [2], the higher Hopf formulae have been studied in [3].

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Milnor fiber complexes and some representations

Alexander R. Miller

H. O. Foulkes discovered some amazing characters for the symmetric group S_n by summing Specht modules of certain ribbon shapes according to height [7]. These characters have some remarkable properties and have been the subject of many investigations, most recently because of connections with adding random numbers, shuffling cards, the Veronese embedding, and combinatorial Hopf algebras, see [2, 5, 6, 9, 17]. We give a new approach to these characters which works for a wide variety of reflection groups. The approach is geometric and based on an object called the Milnor fiber complex. It gives new results and it unifies, explains, and extends previously known (type A) ones. This work appears in [12, 13, 14, 15].

Coxeter and Shephard groups. Let V be an ℓ-dimensional vector space over $\mathbf{C}$, and let G be a finite group with presentation

$$\langle\, r_1, r_2, \ldots, r_\ell \mid r_i^{p_i} = 1,\ \underbrace{r_i r_j r_i \ldots}_{m_{ij}\text{ terms}} = \underbrace{r_j r_i r_j \ldots}_{m_{ji}\text{ terms}} \quad i \neq j\,\rangle \tag{1}$$

where $p_i \geq 2$, $m_{ij} = m_{ji} \geq 2$, and $p_i = p_j$ when m_{ij} is odd. Write $R = \{r_1, \ldots, r_\ell\}$. Finite Coxeter groups are the ones where each p_i is 2. In general G has a Coxeter-like diagram Γ and a canonical faithful representation $G \subset \mathrm{GL}(V)$ as a *(complex) reflection group* in which the generators r_i act on V as *reflections* in the sense that they have finite order and the fixed spaces $\ker(1 - r_i)$ are hyperplanes [10]. The group is identified with its canonical representation as a reflection group and called irreducible if it acts irreducibly on V. Being irreducible is equivalent to the diagram having exactly one connected component. Finite groups with presentation (1) were classified in [10]. The irreducible ones are precisely the finite irreducible Coxeter groups and the groups known as Shephard groups (symmetry groups of objects called *regular complex polytopes* [3] studied by Shephard and Coxeter).

Milnor fiber complex. Associated to G is an abstract simplicial complex Δ with simplices (labeled by) cosets $g\langle I\rangle$ of standard parabolic subgroups $\langle I\rangle$ ($I \subset R$) with face relation "$g\langle I\rangle$ is a face of $h\langle J\rangle$" $\Leftrightarrow$ $g\langle I\rangle \supset h\langle J\rangle$, and with G acting by left translation. If G is a Coxeter group, then this is the classical abstract description of the Coxeter complex [26]. See [22, 19, 12, 15] for details, geometry, and history.

Foulkes characters. Each type-selected subcomplex Δ_S ($S \subset R$) is a bouquet of spheres, and we call the $\mathbf{C}G$-module on the top reduced homology group $H_{|S|-1}(\Delta_S)$ a *ribbon representation*, see [12]. Its character ρ_S is an alternating sum of characters induced by principal characters of parabolic subgroups [12]. The *(generalized) Foulkes characters* defined in [13] are

$$\phi_k = \sum_{\substack{S \subset R \\ |S| = k}} \rho_S \qquad (k = 0, 1, \ldots, \ell). \tag{2}$$

An immediate benefit of this approach is the following formula [13, Theorem 1]

$$\phi_k(g) = \sum_{i=0}^{\ell} (-1)^{k-i} \binom{\ell - i}{k - i} f_{i-1}(\Delta^g) \tag{3}$$

where $\Delta^g = \{\sigma \in \Delta : g\sigma = \sigma\}$ and $f_k(\Sigma)$ is the number of k-simplices in Σ. The face numbers $f_k(\Sigma)$ can be computed with a formula of Orlik and Solomon. Assume G irreducible. Let L be the set of all intersections of reflecting hyperplanes ordered by reverse inclusion, and let μ be the Möbius function. For $X \in L$ define $B_X(t) = (-1)^{\dim X} \sum_{Y \geq X} \mu(X, Y)(-t)^{\dim Y}$. Let $d_1 \leq d_2 \leq \ldots \leq d_\ell$ be the basic degrees of G. Then Orlik [22] (after Orlik–Solomon in the Coxeter case) proved

$$f_{i-1}(\Delta^g) = \sum\nolimits_Y B_Y(d_1 - 1) \tag{4}$$

where the sum is over all i-dimensional subspaces Y above $V^g = \ker(1-g)$ in L.

Elucidating and generalizing classical (type A) results. Our approach elucidates and extends the type A theory (due to Foulkes, Kerber–Thürlings, Diaconis–Fulman, and Isaacs), which previously rested on ad hoc proofs by induction. See [13]. For example, if G is the wreath product $Z_r \wr S_n$ (Z_r cyclic of order r), then L is a Dowling lattice and the restrictions L^X depend only on the dimension of $X \in L$, so that by (3) and (4) the ϕ_i's *depend only on fixed-space dimension* in the sense that $\phi_i(g) = \phi_i(h)$ whenever $\dim V^g = \dim V^h$. The $r = 1$ case of this is the classical fact that the Foulkes characters $\phi_i(g)$ of S_n depend only on the number of cycles of g. The only previous proof of this for S_n is the original one due to Foulkes [7] which uses the Murnaghan–Nakayama rule and induction.

Adding random numbers. Interestingly, these generalized Foulkes characters have recently been connected to adding random numbers in other number systems. Persi Diaconis and Jason Fulman [6] connected the hyperoctahedral ones (type B) to adding random numbers in balanced ternary and other number systems that minimize carries, and Nakano–Sadahiro [16] connected the Foulkes characters for $Z_r \wr S_n$ to a generalized carries process and riffle shuffles.

New phenomena. If G is the wreath product $Z_r \wr S_n$, then the Foulkes characters form a basis for the space of class functions $\chi(g)$ of G that depend only on length $\ell(g) = \min\{k : g = t_1 t_2 \dots t_k,\ t_i \text{ a reflection}\}$, see [13, 14]. Danny Goldstein, Robert M. Guralnick, and Eric M. Rains together made the remarkable experimental observation [18] that in fact the hyperoctahedral Foulkes characters play the role of irreducibles among the hyperoctahedral characters that depend only on length, in the sense that the characters of the hyperoctahedral group B_n that depend only on length are precisely the $\mathbf{N}$-linear combinations of the hyperoctahedral Foulkes characters. We prove this conjecture in [14]. In fact we prove that the same is true for all wreath products $Z_r \wr S_n$ with $r > 1$, not just $r = 2$.

It is an open problem to give a nice description of the characters $\chi(g)$ for S_n ($r = 1$) that depend only on $\ell(g)$, or in other words, that depend only on the number of cycles of g. Kerber [9, p. 306] noticed that already for S_5 the $\mathbf{N}$-linear combinations of Foulkes character do not account for all the characters of S_5 that depend only on length. In [14] we prove that this is always the case for symmetric groups S_n with $n \geq 3$. Note: This line of investigation makes sense for any finite group with given set of generators closed under conjugation.

Curious classification. In [13] we determined all the irreducible cases of G where the ϕ_i's depend only on fixed-space dimension. This led to a curious classification with 11 equivalent conditions [13, Thm. 14]. For example, we find that the ϕ_i's depend only on fixed-space dimension if and only if the sequence of basic degrees $d_1, d_2, \dots, d_\ell$ is arithmetic. Another equivalent condition is that the diagram of G contains no subdiagram of type D_4, F_4, or H_4. We recently found this condition in [1] Abramenko's answer to a geometric problem: In which Coxeter complexes Δ are all walls Δ^r (r a reflection) Coxeter complexes? In [15] we extend Abramenko's result to Milnor fiber complexes in two ways and find another equivalent condition for the Foulkes characters to depend only on fixed-space dimension. In the course of that work we also discovered a beautiful enumerative condition [15, Thm. 11]: if G is irreducible, then the diagram contains no subdiagram of type D_4, F_4, or H_4 if and only if for each $g \in G$ the number of top cells in Δ^g is given by

$$f_{p-1}(\Delta^g) = d_1 d_2 \cdots d_p, \quad p = \dim V^g. \tag{5}$$

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Problem Session

Nate Harman, Aurélien Djament, Roberto Pagaria, Jeremy Miller, Weiyan Chen, Jesse Wolfson, Masahiko Yoshinaga, Alexander R. Miller, Graham Denham, Dan Petersen, Michael Falk

We hosted two evening problem sessions during the workshop. Various participants from very different backgrounds proposed open questions to the audience. The sessions stimulated active discussions among the participants. The problems are collected below in the order that they were proposed.

1. **Nate Harman (University of Chicago).** The following standard theorem from representation theory of the symmetric groups roughly says that "FI-modules see all representations of polynomial growth":

Theorem 1. *Suppose V_n is a sequence of irreducible representations of S_n. If there exists a constant d such that $\dim V_n < n^d$ for all $n \gg 0$, then either V_n or $V_n \otimes \mathrm{sign}$ is a factor of an FI-module generated in degree at most d.*

With the motivation to understand low dimensional representations of the braid group, we ask the following question:

Question 1. *Is there an analog if we replace S_n by the braid group B_n?*

Conjecture 1. *All representations of B_n with slow growth come from finitely generated modules over certain category.*

Ivan Marin remarked that the conjecture is known for linear growth, *e.g.*, in the case when the dimension is $n-1$. (see [5])

2. **Aurélien Djament (CNRS, Nantes).** Let k be a maximal ordered field (*e.g.*, $k = \mathbb{R}$). Let's consider the following monomorphisms between orthogonal groups, for all n and i:

$$O_n(k) \times O_i(k) \hookrightarrow O_{n,i}(k).$$

Question 2. *Does this map induces an isomorphism of $H_d(-, \mathbb{Z})$ for $n \gg d, i$?*

The homology here is understood as the group homology of discrete groups.

For $i = 1$, a theorem of Bökstedt, Brun, and Dupont [2] shows that the answer is yes for $d < n$.