EIGENSPACE ARRANGEMENTS OF REFLECTION GROUPS

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ABSTRACT. The lattice of intersections of reflecting hyperplanes of a complex reflection group W may be considered as the poset of 1-eigenspaces of the elements of W. In this paper we replace 1 with an arbitrary eigenvalue and study the topology and homology representation of the resulting poset. After posing the main question of whether this poset is shellable, we show that all its upper intervals are geometric lattices, and then answer the question in the affirmative for the infinite family $G(m, p, n)$ of complex reflection groups, and the first 31 of the 34 exceptional groups, by constructing CL-shellings. In addition, we completely determine when these eigenspaces of W form a $K(\pi, 1)$ (resp. free) arrangement.

For the symmetric group, we also extend the combinatorial model available for its intersection lattice to all other eigenvalues by introducing *balanced partition posets*, presented as particular upper order ideals of Dowling lattices, study the representation afforded by the top (co)homology group, and give a simple map to the posets of pointed d-divisible partitions.

1. MAIN QUESTION AND RESULTS

Let V denote an n-dimensional $\mathbb{C}$-vector space. A *reflection* in V is any non-identity element r in $\mathrm{GL}(V)$ of finite order that fixes some hyperplane H_r, and a finite subgroup W of $\mathrm{GL}(V)$ is called a *reflection group* if it is generated by reflections.

Example. The action of the symmetric group $\mathfrak{S}_n$ on $[n] := \{1, 2, \ldots, n\}$ gives rise to a faithful action on $\mathbb{C}^n$ via $\sigma(e_i) = e_{\sigma(i)}$, where $e_1, e_2, \ldots, e_n$ denote the standard basis vectors. Since the transpositions act as reflections and generate the group, this representation realizes $\mathfrak{S}_n$ as a reflection group in $\mathrm{GL}(\mathbb{C}^n)$. We shall refer to it as the *defining representation*.

For an element $g \in W$ and root of unity ζ, let $V(g, \zeta)$ denote the ζ-eigenspace of g in V. Define $E(W, \zeta)$ to be the W-poset (partially ordered set) of all such ζ-eigenspaces $\{V(g, \zeta)\}_{g \in W}$ ordered by reverse inclusion, with W-action given by $h \cdot V(g, \zeta) = V(hgh^{-1}, \zeta)$. The eigenvalue $\zeta = 1$ gives the lattice $\mathscr{L}_W$ of intersections of reflecting hyperplanes for W; see [29, Lemma 4.4]. The minimal elements of $E(W, \zeta)$ (i.e., the inclusion-maximal ζ-eigenspaces for W) are the focus of Springer's theory of regular elements [35], and each has dimension equal to the number $a(d)$ of *degrees* $d_1, d_2, \ldots, d_n$ of W that are divisible by d, the order of ζ; see Proposition 3.7 below. When W is crystallographic, the poset $E(W, \zeta)$ itself appears in Broué, Malle, and Michel's Φ-Sylow theory [8].

Example. The degrees of $\mathfrak{S}_4$ are $1, 2, 3, 4$, and those of the dihedral group $I_2(4)$ (whose cardinality is 8) are $2, 4$. Hence the following table.

d	1	2	3	4	≥ 5
$a(d)$ for $\mathfrak{S}_4$	4	2	1	1	0
$a(d)$ for $I_2(4)$	2	2	0	1	0

When $a(d) = 0$, the poset $E(W, \zeta)$ has only one element, the 0-dimensional subspace. For all other listed cases, we have drawn the Hasse diagram of $E(W, \zeta)$ in Figures 1–5 and labeled each eigenspace by linear equations that define it. Maximal eigenspaces are numbered for easy reference. For example, the maximal eigenspace E_4 labeled ④ in Figure 3 is the ζ-eigenspace for the 3-cycle permutation $g = (1, 4, 3)$. Note that the $d = 1, 2$ cases for $I_2(4)$ coincide, since the scalar matrix -1 is an element of $I_2(4)$; see Corollary 3.8 below.
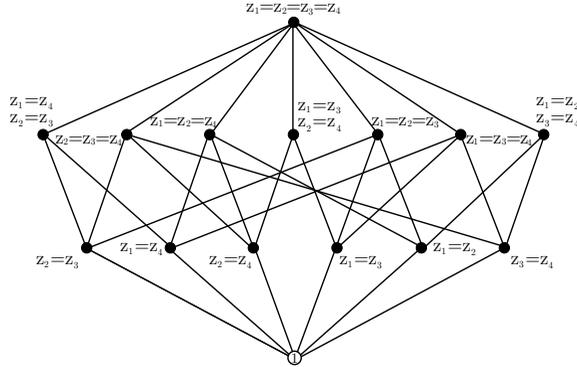
FIGURE 1. The poset $E(\mathfrak{S}_4, 1) = \mathscr{L}_{\mathfrak{S}_4}$ of 1-eigenspaces for $\mathfrak{S}_4 \subset \mathrm{GL}(\mathbb{C}^4)$.
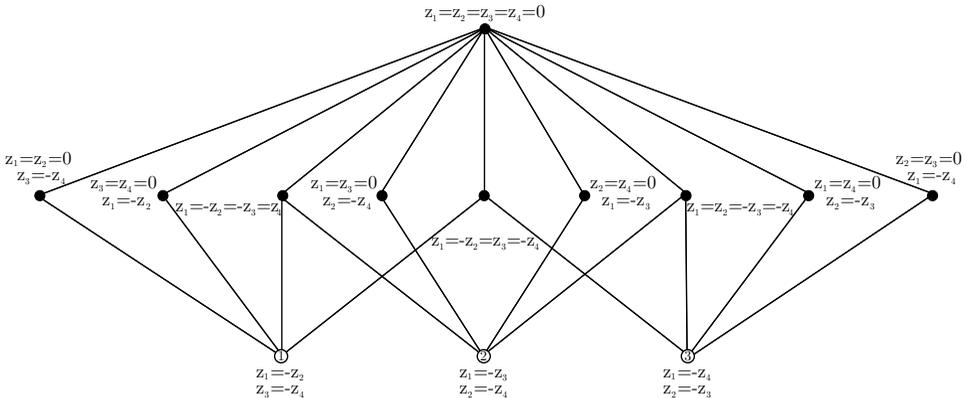
FIGURE 2. The poset $E(\mathfrak{S}_4, -1)$ of (-1)-eigenspaces for $\mathfrak{S}_4 \subset \mathrm{GL}(\mathbb{C}^4)$.
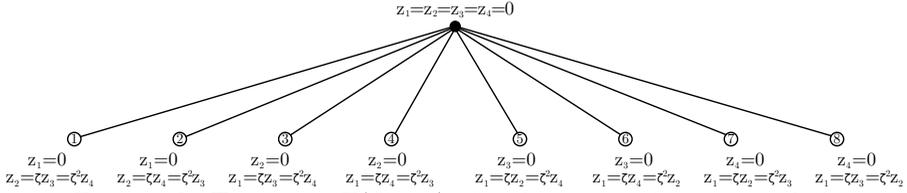
FIGURE 3. The poset $E(\mathfrak{S}_4, \zeta_3)$ of ζ-eigenspaces for ζ any primitive 3rd root of unity and $\mathfrak{S}_4 \subset \mathrm{GL}(\mathbb{C}^4)$.
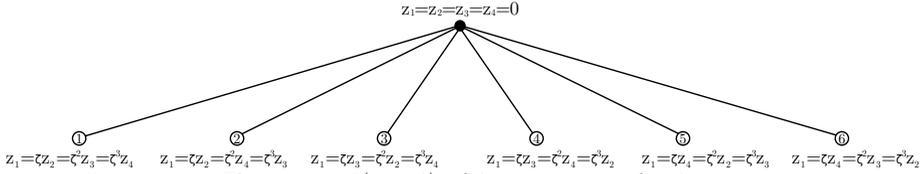
FIGURE 4. The poset $E(\mathfrak{S}_4, \zeta)$ of ζ-eigenspaces for ζ any primitive 4th root of unity and $\mathfrak{S}_4 \subset \mathrm{GL}(\mathbb{C}^4)$.
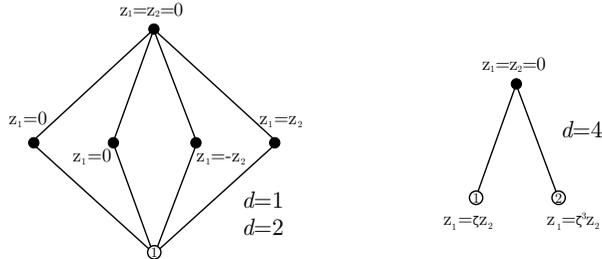
FIGURE 5. The poset $E(W, \zeta)$ of ζ-eigenspaces for the dihedral group $I_2(4)$ of order 8, and ζ of order $d = 1, 2, 4$.

This paper concerns the following problem of Lehrer and Taylor [25, Problem 7].

Problem (Lehrer–Taylor). Study connections between the structure and representations of W and the topology of the posets $E(W, \zeta)$.

In the case of $E(\mathfrak{S}_n, 1)$, Stanley [38] used the work of Hanlon [17] to obtain an explicit expression for the top homology representation, which combined with Klyachko's work to establish a connection with the *Lie representation* Lie_n; see [21, 19]. Lehrer and Solomon [22] extended Stanley's result and conjectured an analogue for all other finite Coxeter groups. Hanlon's work [18] on *Dowling lattices* gives an alternate extension and provides the top homology character of $E(W, 1)$ for $W = G(m, 1, n)$, the complex reflection group of $n \times n$ monomial matrices whose nonzero entries are m^{th} roots of unity. N. Bergeron [1] gave a type B analogue of the above-mentioned Lie correspondence that was later generalized to Dowling lattices by Gottlieb and Wachs [16]. In addition to the above, analogous results have been obtained for various subposets of $E(\mathfrak{S}_n, 1)$; see [45]. However, the author is unaware of any analogous results for $\zeta \neq 1$, even for $\mathfrak{S}_n$.

Question A. Is the following true for every ζ and every reflection group W?

  (Weak version) $E(W, \zeta)$ is homotopy Cohen–Macaulay.

  (Strong version) $\widehat{E(W, \zeta)}$ is CL-shellable.

The fact that CL-shellability implies homotopy CM-ness is well known; see §2.

   Our first main result answers affirmatively the strong version of Question A for all irreducible complex reflection groups except types E_6, E_7, E_8 (which, in the Shephard–Todd classification, are G_{35}, G_{36}, G_{37}). Since the question reduces (see §3 below) to the case where W acts irreducibly, only these three Weyl groups remain.

Theorem 1.1. *Let $W = G(m, p, n)$ or one of the first 31 exceptional groups $G_4, G_5, \ldots, G_{34}$, and let ζ be a primitive d^{th} root of unity. Then $\widehat{E(W, \zeta)}$ is CL-shellable. In particular, the order complex $\Delta(\overline{E(W, \zeta)})$ is a pure bouquet of spheres.*

   Our second main result is central to the first.

Theorem 1.2. *Let W be a reflection group and let ζ be a primitive d^{th} root of unity. Then*

$$(1) \qquad E(W, \zeta) = \{E \cap X \ : \ X \in \mathscr{L}_W \text{ and } E \in E(W, \zeta) \text{ maximal}\}.$$

In particular, each upper interval $[x, \hat{1}]$ in $E(W, \zeta)$ is a geometric lattice.

We will see in Section 3 below that Theorem 1.2 also has interesting consequences of its own, namely, that $E(W, \zeta)$ depends only on d (see Theorem 7.1 for a much sharper result) and is built from copies (conjugates) of the intersection lattice $\mathscr{L}_{W(d)}$ of Lehrer and Springer's reflection subquotient $W(d)$ (see Proposition 3.9).

Remark. The arrangement of ζ-eigenspaces is not always so well behaved for non-reflection groups $G \subset \mathrm{GL}(\mathbb{C}^n)$. For example, take G to be $\mathbb{Z}/3\mathbb{Z} \times \mathfrak{S}_4$ acting faithfully on $\mathbb{C}^4$ by $(i, \sigma)(e_j) = \zeta^i e_{\sigma(j)}$ for some primitive cube root of unity ζ. Then the set of all 1-eigenspaces for G is just the union of the 1-, ζ-, and ζ^2-eigenspaces for the defining representation of $\mathfrak{S}_4$ that we wrote down in Figure 1 and Figure 3 above. The end result is the Hasse diagram of $E(G, 1)$ in Figure 6, which shows that for this non-reflection group $G \subset \mathrm{GL}(\mathbb{C}^4)$, the poset of 1-eigenspaces $E(G, 1)$ is not Cohen–Macaulay, and its upper intervals are not all geometric lattices.
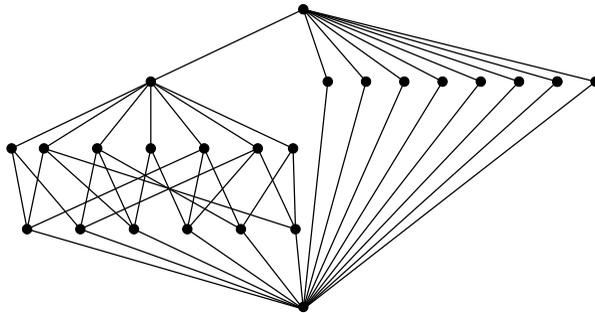
FIGURE 6. The poset of 1-eigenspaces for the group G in Remark that falsifies the conclusion of both Theorem 1.1 and Theorem 1.2 for a non-reflection group.

We also characterize when an eigenspace arrangement of a reflection group comes from an arrangement of hyperplanes (Lemma 10.3 below). As a side benefit, this will answer the natural $K(\pi, 1)$ and freeness questions for eigenspaces, that is, we determine exactly when the complement of the proper ζ-eigenspaces of a reflection group W is $K(\pi, 1)$, and exactly when the ζ-eigenspaces of codimension 1 form a free (hyperplane) arrangement. For brevity, define $\mathcal{A}(W, d)$ to be the set of all *proper ζ-eigenspaces of the reflection group W that are maximal under inclusion*, for a fixed but arbitrary choice of primitive d^{th} root of unity ζ, so that the complement of the proper ζ-eigenspaces of W is

$$(2) \qquad \mathcal{M}(W, d) = \mathbb{C}^n \setminus \bigcup_{X \in \mathcal{A}(W,d)} X.$$

The two questions were resolved affirmatively for the highly nontrivial case of the *reflection arrangement* $\mathcal{A}(W, 1)$ by Bessis and Terao, respectively, and in contrast to Question A, the non-classical case $\mathcal{A}(W, d) \neq \mathcal{A}(W, 1)$ is surprisingly simple.

Theorem 1.3. *Let $W \subset \mathrm{GL}(\mathbb{C}^n)$ be a complex reflection group and let $d > 0$. Then the following are equivalent:*
  (i) One has $a(d) = n$ or $a(d) = n - 1$. That is, $\mathcal{A}(W, d)$ contains a hyperplane.
  (ii) The arrangement $\mathcal{A}(W, d)$ is a free hyperplane arrangement.
  (iii) The complement $\mathcal{M}(W, d)$ is a $K(\pi, 1)$ space.
Furthermore, $\mathcal{A}(W, d) = \mathcal{A}(W, 1)$ if and only if $a(d) = n$.

The reader should be warned that no *new* examples of free or $K(\pi, 1)$ arrangements occur in Theorem 1.3. Nevertheless, the result is essentially all one could hope for, and gives a nice extension of the classical picture to arbitrary eigenvalues.

Contents

2. Preliminaries

Recall that a *G-poset* is a poset P with a G-action that preserves order, i.e., $x < y$ implies $gx < gy$ for all $g \in G$ and $x, y \in P$. It is *bounded* if it has a unique minimal element (called the *bottom element* and denoted by $\hat{0}$) and a unique maximal element (called the *top element* and denoted by $\hat{1}$). Write $P \cup \{\hat{0}\}$ for the poset obtained from P by adjoining a new element $\hat{0}$, regardless of whether P has a bottom element. Similarly, $P \cup \{\hat{1}\}$ is obtained by adjoining a new element $\hat{1}$. Appending both yields $\widehat{P} := P \cup \{\hat{0}, \hat{1}\}$.

The *order complex* of a poset P is the (abstract) simplicial complex $\Delta(P)$ consisting of all totally ordered sets $x_1 < x_2 < \ldots < x_i$ in P. Because it has a cone point (and is therefore contractible) if P has a top or bottom element, one often considers the *proper part* $\overline{P} := P \setminus \{\hat{0}, \hat{1}\}$ of the poset, which is just P if neither a top nor bottom element is present.

Recall that a (finite) simplicial complex Δ is *k-connected* if its homotopy groups $\pi_j(\Delta)$ are trivial for $0 \le j \le k$. Define the *link* of a face $F \in \Delta$ to be the subcomplex

$$\mathrm{lk}_\Delta(F) = \{G \in \Delta \ : \ G \cup F \in \Delta, \ G \cap F = \varnothing\},$$

and say Δ is *homotopy Cohen–Macaulay* (abbreviated to HCM) if for each face $F \in \Delta$ the link $\mathrm{lk}_\Delta(F)$ is $(\dim \mathrm{lk}_\Delta(F) - 1)$-connected. For $\mathbf{k}$ a field or $\mathbb{Z}$, the simplicial complex is said to be *Cohen–Macaulay* over $\mathbf{k}$ (abbreviated to CM/$\mathbf{k}$, or just CM when $\mathbf{k} = \mathbb{Z}$) if for each face $F \in \Delta$ the homology groups $\widetilde{H}_i(\mathrm{lk}_\Delta(F); \mathbf{k})$ vanish for $i < \dim \mathrm{lk}_\Delta(F)$. The property of being Cohen–Macaulay is invariant under homeomorphisms and implies that the complex is homologically a bouquet of $(\dim \Delta)$-spheres, whereas the stronger property of being homotopy Cohen–Macaulay is not invariant under homeomorphisms but implies that the complex is homotopically a bouquet of $(\dim \Delta)$-spheres; see [27] and [32, p. 117], respectively, or surveys [4, 45].

Though there are many techniques for establishing Cohen–Macaulayness, we shall be concerned with *(pure) CL-shellability*. A simplicial complex Δ which is pure d-dimensional (i.e., each maximal face under inclusion has dimension d) is said to be *shellable* if its maximal faces (called *facets*) can be ordered $F_1, F_2, \ldots, F_\ell$ so that for each k the subcomplex generated by the first k facets intersects the $(k+1)$st facet in a pure $(d-1)$-dimensional subcomplex. A poset P is called *shellable* (resp. *HCM, CM*) if its order complex $\Delta(P)$ is shellable (resp. HCM, CM). Finally, a *CL-shellable*[1] poset is a bounded poset that *admits a recursive atom ordering*, as defined in Section 4. The following implications for poset shellability are strict:

$$\text{CL-shellable} \Rightarrow \text{shellable} \Rightarrow \text{HCM} \Rightarrow \text{CM} \Rightarrow \text{CM/}\mathbf{k} \Rightarrow \text{CM/}\mathbb{Q},$$

for any field $\mathbf{k}$ of positive characteristic. Note that a poset P is shellable (resp. HCM, CM) if and only if $\overline{P}$, or just $P \setminus \{\hat{1}\}$, is shellable (resp. HCM, CM).

3. General reductions

Shephard and Todd classified all irreducible reflection groups in [34]. There are 34 exceptional groups in their classification, labeled $G_4, G_5, \ldots, G_{37}$, and 3 infinite families, explained below:

- $W(A_n) \subset \mathrm{GL}(\mathbb{C}^n)$.

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[1]This recursive formulation of CL-shellability is due to Björner and Wachs [5].

- $G(m, p, n)$ with $m > 1$, p a divisor of m, and $(m, p, n) \neq (2, 2, 2)$.
- $C_m := G(m, 1, 1)$.

$W(A_n)$ denotes the representation of $\mathfrak{S}_{n+1}$ one obtains from the defining representation of §1 after modding out by the fixed space $\mathbb{C}(e_1 + e_2 + \ldots + e_{n+1})$. For $\boldsymbol{\mu}_m$ the set of mth roots of unity (and p a divisor or m), the group $G(m, p, n)$ consists of all $n \times n$ monomial matrices with nonzero entries in $\boldsymbol{\mu}_m$ whose product lies in $\boldsymbol{\mu}_{m/p}$. General reflection groups decompose into irreducible ones as follows.

Proposition 3.1 (Theorem 1.27 in [25])**.** *Let $W \subset \mathrm{GL}(V)$ be a reflection group. Let $V_1, \ldots, V_k$ denote the nontrivial irreducible submodules of V, so that the restriction W_i of W to V_i is irreducible. Then $W \cong W_1 \times \cdots \times W_k$ and one has an orthogonal sum decomposition $V = V^W \oplus V_1 \oplus \ldots \oplus V_k$ with respect to a chosen W-invariant positive definite Hermitian form.*

Corollary 3.2. *Maintain the notation of Proposition 3.1 and let ζ be a root of unity. Then*

$$(3) \qquad\qquad E(W, \zeta) \cong E(W_1, \zeta) \times \cdots \times E(W_k, \zeta).$$

In particular, the family of posets $E(W, \zeta)$ obtained by letting W vary over the three infinite families above is the same as that obtained by letting W vary over the single infinite family $G(m, p, n)$. Another consequence is that Question A and Theorem 1.2 reduce to irreducible reflection groups.

Corollary 3.3. *Maintain the notation of Corollary 3.2. Then the following hold.*
  (i) $E(W, \zeta)$ is HCM (resp. CM) if and only if each $E(W_i, \zeta)$ is HCM (resp. CM).
  (ii) $\widehat{E(W, \zeta)}$ is CL-shellable if and only if each $\widehat{E(W_i, \zeta)}$ is CL-shellable.

Proof. Note that each $E(W_i, \zeta)$ has a top element $\hat{1}_i$. Then (i) follows from (3) and a homeomorphism of Quillen (see [32, Ex. 8.1] and [46, Thm. 5.1(b)]), while (ii) follows from (3) and [6, Thm. 10.16]. □

Because of the important role that *maximal* eigenspaces play in what follows, we make the following convention before proceeding.

Convention 3.4. A *maximal* ζ-eigenspace for W is one that is not properly contained in any other. Because such a space is minimal with respect to the poset order of $E(W, \zeta)$ given by reverse inclusion, in order to avoid confusion we shall always take *minimal* and *maximal* to be with respect to inclusion when dealing with subspaces. For example, "$E \in E(W, \zeta)$ maximal" thus means that E is not properly contained in any $V(g, \zeta)$.

A reflection group $W \subset \mathrm{GL}(V)$ acts on the algebra of polynomial functions $S = S(V^*)$ via $gf(v) = f(g^{-1}v)$, and Shephard and Todd showed that the subalgebra S^W of W-fixed polynomials is again polynomial, generated by $n := \dim V$ algebraically independent homogeneous polynomials $f_1, f_2, \ldots, f_n$, called *basic invariants*, the degrees of which are uniquely determined by the group W and denoted by $d_1 \leq d_2 \leq \ldots \leq d_n$. We shall always assume an indexing such that $\deg(f_i) = d_i$. For $d > 0$, write $a(d)$ for the number of d_i divisible by d as in §1; see Springer [35].

Proposition 3.5 (Springer)**.** *Let W be a reflection group, let ζ be a primitive d^{th} root of unity, and let $f_1, f_2, \ldots, f_n$ be a set of basic invariants for W. Set $H_i := \{v \in V \ : \ f_i(v) = 0\}$. Then one has $\bigcup_{g \in W} V(g, \zeta) = \bigcap_{d \nmid d_i} H_i$.*

As a consequence, the collection of maximal ζ-eigenspaces of W depends only on (the group and) the multiset of degrees d_i that are divisible by d, which we shall denote by $A(d) := \{d_i \; : \; d \mid d_i\}$, so that $a(d) = |A(d)|$; see Theorem 7.1 below for a sharper result.

Corollary 3.6. *Let W be a reflection group. Let ζ and ζ' be roots of unity of orders d and d' such that $A(d) = A(d')$. Then the set of maximal ζ-eigenspaces of W coincides with the set of maximal ζ'-eigenspaces of W.*

Proposition 3.7 (Springer)**.** *Let W be a reflection group, let ζ be a primitive d^{th} root of unity, and let $E, E' \in E(W, \zeta)$ be maximal eigenspaces. Then*

(i) $\dim E = a(d)$, *and*
(ii) $gE = E'$ *for some $g \in W$.*

Corollary 3.8. *Let W be a reflection group and let ζ be a primitive d^{th} root of unity. Then the following are equivalent.*

(i) $E(W, \zeta) = \mathscr{L}_W$.
(ii) $\mathbb{C}^n \in E(W, \zeta)$.
(iii) $a(d) = n$.

Proof. Clearly (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii). Assume $\mathbb{C}^n \in E(W, \zeta)$ so that $\zeta \in W$, and hence $\zeta^{-1}W = W$. Writing $V(g, \zeta) = V(\zeta^{-1}g, 1)$ for each $g \in W$, it follows that $E(W, \zeta) = \mathscr{L}_W$. $\qquad\square$

Theorem 3.9 (Lehrer–Springer [23, 24])**.** *Let W be a reflection group, let ζ be a primitive d^{th} root of unity, and let $E \in E(W, \zeta)$ be maximal with normalizer*

$$N_W(E) = \{g \in W \; : \; gE \subseteq E\}$$

and centralizer

$$Z_W(E) = \{g \in W \; : \; gv = v \text{ for all } v \in E\}.$$

Then $\overline{N} := N_W(E)/Z_W(E)$ acts as a reflection group on E, and the following hold:

(i) *If $f_1, f_2, \ldots, f_n$ form a set of basic invariants for W, then the restrictions $f_i|_E$ of those whose degree d_i is divisible by d form a set of basic invariants for $\overline{N}$.*
(ii) *The reflecting hyperplanes of $\overline{N}$ on E are the intersections of E with the reflecting hyperplanes of W that do not contain E.*
(iii) *If W is irreducible, then $\overline{N}$ acts irreducibly on E.*
(iv) *$W(d) := \overline{N}$ is uniquely determined by W and d, up to conjugation by W.*

Proposition 3.10. *Let W be a reflection group and let ζ be a root of unity. Then one has an inclusion*

$$(4) \qquad E(W, \zeta) \subseteq \{E \cap X \; : \; X \in \mathscr{L}_W, \; E \in E(W, \zeta) \text{ maximal}\}$$

and the following are equivalent:

(i) *Equality in (4).*
(ii) *There exists $E \in E(W, \zeta)$ maximal such that for all $X \in \mathscr{L}_W$ one has $E \cap X \in E(W, \zeta)$.*
(iii) *For every $E \in E(W, \zeta)$ maximal and $X \in \mathscr{L}_W$ one has $E \cap X \in E(W, \zeta)$.*
(iv) *There exists $E \in E(W, \zeta)$ maximal such that $[E, \hat{1}] = \mathscr{L}_{N_W(E)/Z_W(E)}$.*
(v) *For every $E \in E(W, \zeta)$ maximal, one has $[E, \hat{1}] = \mathscr{L}_{N_W(E)/Z_W(E)}$.*

Proof. For the inclusion (4), let $Y \in E(W, \zeta)$ and choose a maximal ζ-eigenspace E such that $Y \subseteq E$. Write $E = V(g, \zeta)$ and $Y = V(h, \zeta)$ for some $g, h \in W$. Then $v \in Y$ if and only if $hv = \zeta v = gv$, i.e., if and only if $v \in E \cap V(g^{-1}h, 1)$. Hence $Y = E \cap X$ for some $X \in \mathscr{L}_W$. As for the equivalences, clearly (i) is equivalent to (iii), which is equivalent to (v) by Theorem 3.9(ii), and the remaining two equivalences (ii)$\Leftrightarrow$(iii) and (iv)$\Leftrightarrow$(v) follow from Proposition 3.7(ii). $\qquad\square$

4. $G(m, p, n)$ CASE OF THEOREM 1.2

Recall that $\boldsymbol{\mu}_m$ denotes the collection of all m^{th} roots of unity, and that $G(m, p, n)$ denotes the group of all $n \times n$ monomial matrices with nonzero entries in $\boldsymbol{\mu}_m$ whose product lies in $\boldsymbol{\mu}_{m/p}$. Note that $G(1, 1, n)$ is the defining representation of $\mathfrak{S}_n$ given in §1, and that $G(1, 1, n) \subseteq G(m, p, n) \subseteq G(m, 1, n)$. When $m > p$ the set of reflecting hyperplanes for $G(m, p, n)$ coincides with that for $G(m, 1, n)$ and is given by the union of the following two sets:

$$(5) \qquad \{z_i = \xi z_j \ : \ 1 \le i < j \le n, \ \xi \in \boldsymbol{\mu}_m\}$$

$$(6) \qquad \{z_i = 0 \ : \ 1 \le i \le n\}.$$

When $m = p$ the set of reflecting hyperplanes for $G(p, p, n)$ is just given by (5).

For roots of unity $\delta_1, \delta_2, \ldots, \delta_\ell \in \boldsymbol{\mu}_m$ and an ℓ-set $\{\sigma_1, \sigma_2, \ldots, \sigma_\ell\} \subseteq [n]$, identify the 2-line array

$$\sigma = \begin{pmatrix} \sigma_1 & \sigma_2 & \ldots & \sigma_\ell \\ \delta_1\sigma_2 & \delta_2\sigma_3 & \ldots & \delta_\ell\sigma_1 \end{pmatrix}$$

with the linear map that fixes each e_i with $i \in [n] \setminus \{\sigma_1, \ldots, \sigma_\ell\}$ and that sends e_{σ_i} to $\delta_i e_{\sigma_{i+1}}$ for $i \in [\ell - 1]$, while $e_{\sigma_\ell} \mapsto \delta_\ell e_{\sigma_1}$. Because multiple arrays may represent the same map, in the next section we will require that $\sigma_1 < \sigma_2, \sigma_3, \ldots, \sigma_\ell$, but we postpone the restriction until then. Call such an element σ a *(colored) cycle*, and define

- $\ell(\sigma) := \ell$ (the *length* of σ),
- $\text{Supp}(\sigma) := \{\sigma_1, \ldots, \sigma_\ell\}$ (the *support* of σ), and
- $\text{Col}(\sigma) := \{\delta_1, \ldots, \delta_\ell\}$ (the *multiset of colors* of σ).

With two cycles σ, σ' said to be *disjoint* if $\text{Supp}(\sigma) \cap \text{Supp}(\sigma') = \varnothing$, note that any element of $G(m, 1, n)$ can be decomposed into a product $\sigma^{(1)}\sigma^{(2)} \cdots \sigma^{(q)}$ of disjoint cycles, and that such a product is an element of $G(m, p, n)$ if and only if $\prod_i \prod_{\delta \in \text{Col}(\sigma^{(i)})} \delta$ is an element of $\boldsymbol{\mu}_{m/p}$. The following lemma is a straightforward calculation.

Lemma 4.1. *Let $\sigma \in G(m, 1, n)$ be a cycle and write $\sigma = \begin{pmatrix} \sigma_1 & \sigma_2 & \ldots & \sigma_\ell \\ \delta_1\sigma_2 & \delta_2\sigma_3 & \ldots & \delta_\ell\sigma_1 \end{pmatrix}$. Let $\zeta \neq 1$ be a root of unity. Then*

$$\dim V(\sigma, \zeta) = \begin{cases} 1 & \text{if } \zeta^\ell = \prod \delta_i; \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, in the former case $V(\sigma, \zeta)$ is the solution set of the following equations:

$$(7) \qquad z_{\sigma_1} = \zeta\delta_1^{-1}z_{\sigma_2} = \zeta^2(\delta_1\delta_2)^{-1}z_{\sigma_3} = \ldots = \zeta^{\ell-1}(\delta_1\delta_2\cdots\delta_{\ell-1})^{-1}z_{\sigma_\ell}$$

$$(8) \qquad z_i = 0 \quad \text{for} \quad i \in [n] \setminus \{\sigma_1, \sigma_2, \ldots, \sigma_\ell\}.$$

The crux of Theorem 1.2 is Proposition 4.3 below, for which we will need the following.

Lemma 4.2. *Let $W = G(m, p, n)$, let $g \in W$, and let ζ be a primitive d^{th} root of unity. Suppose that $d \mid m$ and that $a(d) < n$. Then there exists an $i \in [n]$ such that for all $\mathbf{z} \in V(g, \zeta)$ one has $z_i = 0$.*

Proof. It suffices to assume that $V(g, \zeta)$ is maximal. Since $\zeta \in \boldsymbol{\mu}_m$, we have that W contains

$$h := \begin{pmatrix} 1 \\ \zeta 1 \end{pmatrix} \begin{pmatrix} 2 \\ \zeta 2 \end{pmatrix} \cdots \begin{pmatrix} a(d) \\ \zeta a(d) \end{pmatrix} \begin{pmatrix} n \\ \zeta^{-a(d)} n \end{pmatrix},$$

which must have $\dim V(h, \zeta) = a(d)$ by Proposition 3.7(i), and therefore $z_n = 0$ for each $\mathbf{z} \in V(h, \zeta)$. Since W acts transitively on its maximal ζ-eigenspaces by Proposition 3.7(ii), the result follows. $\square$

In the next proposition we define another group W' within the $G(m, p, n)$ family, that contains W. In particular, $\mathscr{L}_W \subset \mathscr{L}_{W'}$. We do so to obtain a stronger version of Theorem 1.2 (Theorem 4.4 below) which will be used in Sections 8 and 9.

Proposition 4.3. *Let $W = G(m, p, n)$, let ζ be a primitive d^{th} root of unity, and let $m \vee d$ denote the least common multiple of m and d. Set*

$$W' := \begin{cases} W & \text{if } a(d) = n; \\ G(m \vee d, 1, n) & \text{if } a(d) < n. \end{cases}$$

Then $V(g, \zeta) \cap X \in E(W, \zeta)$ for every $g \in W$ and $X \in \mathscr{L}_{W'}$.

Proof. If $a(d) = n$, then one has $E(W, \zeta) = E(W, 1) = \mathscr{L}_W$, and the result follows.

Assume that $a(d) < n$. It suffices to show that $V(g, \zeta) \cap H \in E(W, \zeta)$ for every $g \in W$ and every *reflecting hyperplane* H of W', since a general element of $\mathscr{L}_{W'}$ is an intersection of reflecting hyperplanes. Let $g \in W$ and let $r \in W'$ be a reflection with fixed space H. We show that $V(g, \zeta) \cap H$ is a ζ-eigenspace of W by exhibiting an $h \in W$ such that $V(h, \zeta) = V(g, \zeta) \cap H$.

Set $h = g$ if $V(g, \zeta) \subseteq H$. Assume otherwise so that $V(g, \zeta) \not\subseteq H$. Write $g = \sigma^{(1)} \sigma^{(2)} \cdots \sigma^{(q)}$ as a maximal product of nonempty disjoint cycles so that $[n] = \cup_i \mathrm{Supp}(\sigma^{(i)})$ and $V(g, \zeta) = \oplus_i V(\sigma^{(i)}, \zeta)$, and define $\Sigma := \{\sigma^{(1)}, \ldots, \sigma^{(q)}\}$. Since H is of the form $z_j = 0$ or $z_j = \xi z_k$, we see from Lemma 4.1 that it contains all but exactly one or exactly two of the eigenspaces $V(\sigma^{(i)}, \zeta)$.

Case 1. There exists exactly one cycle $\sigma \in \Sigma$ such that $V(\sigma, \zeta) \not\subseteq H$.
Subcase 1a. $\ell(\sigma) \geq 2$.

Write $\sigma = \begin{pmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_{\ell-1} & \sigma_\ell \\ \delta_1 \sigma_2 & \delta_2 \sigma_3 & \cdots & \delta_{\ell-1} \sigma_\ell & \delta_\ell \sigma_1 \end{pmatrix}$ and let h be the element of W that one obtains from g by replacing σ with the product of the cycles

$$\sigma' := \begin{pmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_{\ell-2} & \sigma_{\ell-1} \\ \delta_1 \sigma_2 & \delta_2 \sigma_3 & \cdots & \delta_{\ell-2} \sigma_{\ell-1} & \delta_{\ell-1} \delta_\ell \sigma_1 \end{pmatrix} \quad \text{and} \quad \sigma'' := \begin{pmatrix} \sigma_\ell \\ \sigma_\ell \end{pmatrix}.$$

Then $V(h, \zeta) = V(g, \zeta) \cap H$ if and only if $V(\sigma', \zeta) = V(\sigma'', \zeta) = \{0\}$. Clearly $V(\sigma'', \zeta) = \{0\}$, since $d > 1$. Suppose that $V(\sigma', \zeta) \neq \{0\}$. Since $V(\sigma, \zeta) \neq \{0\}$ also, Lemma 4.1 tells us that $\zeta^\ell = \delta_1 \delta_2 \cdots \delta_\ell = \zeta^{\ell-1}$. However, $\zeta \neq 1$.

For example, if $W = G(2, 2, 8)$ and $\zeta = e^{2\pi i/6}$, then for

$$g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 5 & 6 & 7 \\ -6 & -7 & -5 \end{pmatrix} \begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix}$$
$$H = \{\mathbf{z} \in \mathbb{C}^8 \ : \ z_5 = \zeta^2 z_6\},$$

one has $h = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & 1 \end{pmatrix} \left[\begin{pmatrix} 5 & 6 \\ -6 & -(-5) \end{pmatrix} \begin{pmatrix} 7 \\ 7 \end{pmatrix} \right] \begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix}$.

Subcase 1b. $\ell(\sigma) = 1$.

Then $\sigma = \begin{pmatrix} \tau_0 \\ \zeta\tau_0 \end{pmatrix}$ for some $\tau_0 \in [n]$, implying that $\zeta \in \boldsymbol{\mu}_m$, and so $d \mid m$ and $a(d) < n$. Applying Lemmas 4.1 and 4.2, it follows that $V(\tau, \zeta) = \{0\}$ for some $\tau \in \Sigma$. Note that necessarily $\tau \neq \sigma$, as their ζ-eigenspaces disagree. Write

$$\tau = \begin{pmatrix} \tau_1 & \tau_2 & \cdots & \tau_\ell \\ \epsilon_1\tau_2 & \epsilon_2\tau_3 & \cdots & \epsilon_\ell\tau_1 \end{pmatrix}$$

and obtain h from g by replacing the two cycles σ, τ with the single cycle

$$\sigma' = \begin{pmatrix} \tau_0 & \tau_1 & \tau_2 & \cdots & \tau_\ell \\ \zeta\tau_1 & \epsilon_1\tau_2 & \epsilon_2\tau_3 & \cdots & \epsilon_\ell\tau_0 \end{pmatrix}.$$

Applying Lemma 4.1 shows that $V(\sigma', \zeta) = \{0\}$; indeed, one has $\zeta^\ell \neq \prod_{i=1}^\ell \epsilon_i$, and so $\zeta^{\ell+1} \neq \zeta \prod_{i=1}^\ell \epsilon_i$. It follows that $V(h, \zeta) = V(g, \zeta) \cap H$. That h is in W is clear.

For example, if $W = G(6, 6, 4)$ and $\zeta = e^{2\pi i/3}$, then for $g = \begin{pmatrix} 2 \\ \zeta 2 \end{pmatrix} \begin{pmatrix} 3 \\ \zeta 3 \end{pmatrix} \begin{pmatrix} 4 \\ \zeta 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

and $H = \{\mathbf{z} \in \mathbb{C}^4 \ : \ z_3 = 0\}$, one has $h = \begin{pmatrix} 2 \\ \zeta 2 \end{pmatrix} \begin{pmatrix} 4 \\ \zeta 4 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ \zeta 1 & 3 \end{pmatrix}$.

Case 2. There exist exactly two cycles $\sigma, \tau \in \Sigma$ such that $V(\sigma, \zeta), V(\tau, \zeta) \not\subseteq H$.

Then for some $\xi \in \boldsymbol{\mu}_{m \vee d}$ and a suitable indexing, we have

$$\sigma = \begin{pmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_s \\ \delta_1\sigma_2 & \delta_2\sigma_3 & \cdots & \delta_s\sigma_1 \end{pmatrix}, \tau = \begin{pmatrix} \tau_1 & \tau_2 & \cdots & \tau_t \\ \epsilon_1\tau_2 & \epsilon_2\tau_3 & \cdots & \epsilon_t\tau_1 \end{pmatrix}, r = \begin{pmatrix} \sigma_1 & \tau_1 \\ \xi^{-1}\tau_1 & \xi\sigma_1 \end{pmatrix}$$

Since H is given by $z_{\sigma_1} = \xi z_{\tau_1}$, Lemma 4.1 implies that $(V(\sigma, \zeta) \oplus V(\tau, \zeta)) \cap H$ consists of the points $\mathbf{z} \in \mathbb{C}^n$ that satisfy $z_i = 0$ for $i \notin \text{Supp}(\sigma) \cup \text{Supp}(\tau)$ and

$$(9) \qquad z_{\sigma_1} = \zeta\delta_1^{-1} z_{\sigma_2} = \ldots = \zeta^{s-1}(\delta_1 \cdots \delta_{s-1})^{-1} z_{\sigma_s}$$
$$= \xi z_{\tau_1} = \xi\zeta\epsilon_1^{-1} z_{\tau_2} = \ldots = \xi\zeta^{t-1}(\epsilon_1 \cdots \epsilon_{t-1})^{-1} z_{\tau_t}.$$

Let k be such that the coefficient ϵ of z_{τ_k} in (9) is an element of $\boldsymbol{\mu}_m$. (For existence, note that the cosets $\boldsymbol{\mu}_m, \zeta\boldsymbol{\mu}_m, \ldots, \zeta^{t-1}\boldsymbol{\mu}_m$ cover the group $\boldsymbol{\mu}_d\boldsymbol{\mu}_m$, since ζ generates $\boldsymbol{\mu}_d$ and $\zeta^t \in \boldsymbol{\mu}_m$, then observe that ξ permutes these cosets, since $\boldsymbol{\mu}_{m \vee d} = \boldsymbol{\mu}_m\boldsymbol{\mu}_d$ from basic algebra.) Then

$$(10) \qquad (V(\sigma, \zeta) \oplus V(\tau, \zeta)) \cap H = (V(\sigma, \zeta) \oplus V(\tau, \zeta)) \cap H'$$

for H' the reflecting hyperplane of $r' = \begin{pmatrix} \sigma_1 & \tau_k \\ \epsilon^{-1}\tau_k & \epsilon\sigma_1 \end{pmatrix}$.

We claim that $h := gr'$ satisfies $V(h, \zeta) = V(g, \zeta) \cap H$, or in other words, that $V(\sigma\tau r') = V(\sigma\tau) \cap H$. To see this, employ (10) to rewrite the equality as

$$(11) \qquad V(\sigma\tau r', \zeta) = (V(\sigma, \zeta) \oplus V(\tau, \zeta)) \cap H'.$$

Now observe that, on one hand, $V(\sigma\tau, \zeta) \cap H'$ is clearly contained in $V(\sigma\tau r', \zeta)$ and has dimension 1 by hypothesis. On the other hand, $\sigma\tau r'$ is necessarily a cycle, and therefore has ζ-eigenspace of dimension at most 1 by Lemma 4.1.

For example, if $W = G(6,3,8)$ and $\zeta = e^{2\pi i/9}$, then for $\omega := e^{2\pi i/6}$ and

$$g = \begin{pmatrix} 1 & 2 & 3 \\ \omega 2 & 3 & \omega 1 \end{pmatrix} \begin{pmatrix} 4 & 5 & 6 \\ -5 & \omega^4 6 & \omega 4 \end{pmatrix} \begin{pmatrix} 7 & 8 \\ \omega 8 & \omega 7 \end{pmatrix}$$

$$\xi = e^{2\pi i/18}$$

$$r = \begin{pmatrix} 1 & 4 \\ \xi^{-1}4 & \xi 1 \end{pmatrix},$$

one has $r' = \begin{pmatrix} 1 & 6 \\ \omega^2 6 & \omega^{-2}1 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 4 & 5 & 6 & 2 & 3 \\ -4 & -5 & \omega^4 6 & \omega^5 2 & 3 & \omega 1 \end{pmatrix} \begin{pmatrix} 7 & 8 \\ \omega 8 & \omega 7 \end{pmatrix}$. $\square$

Proof of Theorem 1.2 for G(m, p, n). Note that $\mathscr{L}_W \subseteq \mathscr{L}_{W'}$ in Proposition 4.3, since W is a subgroup of W', then invoke Proposition 3.10. $\square$

Another consequence of Proposition 4.3 is the following stronger result, which will play an important role in Sections 8 and 9 below.

Theorem 4.4. *Let* $W = G(m,p,n)$, *let* ζ *be a primitive* d^{th} *root of unity, and let* W' *be as in Proposition 4.3. Then*

(i) $E(W,\zeta) \subseteq \mathscr{L}_{W'}$, *and*
(ii) $V(g,\zeta) \cap X \in E(W,\zeta)$ *for every* $g \in W$ *and* $X \in \mathscr{L}_{W'}$.
In other words, $E(W,\zeta)$ *is an upper order ideal of* $\mathscr{L}_{W'}$.

Proof. Observe that W' contains the scalar matrix ζ, and that

$$V(g,\zeta) = V(\zeta^{-1}g, 1) \in \mathscr{L}_{W'}$$

whenever $g \in W$. The first claim follows, and the second is Proposition 4.3. $\square$

5. Maximal eigenspaces of $G(m,p,n)$

In this section we associate a certain *word*, denoted by $\mathrm{word}(E)$, to each maximal eigenspace $E \in E(W,\zeta)$ for $W = G(m,p,n)$ and $\zeta \neq 1$. In the next section we show that lexicographically ordering these words gives a recursive atom ordering for $\widehat{E(W,\zeta)}$. In addition to $a(d)$, the following number plays an important role in our discussion.

$$(12) \qquad \ell(d) \stackrel{\mathbf{def}}{=} \min\{s \in \mathbb{Z}_{\geq 1} \ : \ \zeta^s \in \boldsymbol{\mu}_m\}$$

$$(13) \qquad = \frac{d}{\gcd(m,d)}.$$

The crux of our construction is that each maximal eigenspace $E \in E(W,\zeta)$ determines a unique set of $a(d)$ many $\ell(d)$-cycles in $G(m,1,n)$ whose product has ζ-eigenspace E. It is from this set that we construct $\mathrm{word}(E)$ in Corollary 5.4 below. We establish the correspondence by first showing that any product $g \in G(m,1,n)$ of $\dim V(g,\zeta)$ many nontrivial $\ell(d)$-cycles is uniquely determined by its eigenspace $V(g,\zeta)$, and then showing that each maximal $E \in E(W,\zeta)$ may be realized as the ζ-eigenspace of such a product.

Recall from Section 4 the identification of a 2-line array of the form

$$(14) \qquad \sigma = \begin{pmatrix} \sigma_1 & \sigma_2 & \dots & \sigma_\ell \\ \delta_1\sigma_2 & \delta_2\sigma_3 & \dots & \delta_\ell\sigma_1 \end{pmatrix}$$

and a particular element of $G(m,1,n)$, and note that the element determines the 2-line array up to cyclically permuting columns. Thus, by requiring that the smallest

σ_i come first, the array is uniquely determined. We adopt this convention for the remainder of the section, i.e., that $\sigma_1 < \sigma_2, \sigma_3, \ldots, \sigma_\ell$.

Lemma 5.1. *Let ζ be a primitive d^{th} root of unity for $d > 1$, and set $\ell := \ell(d)$. Suppose that $\sigma, \tau \in G(m, 1, n)$ are two ℓ-cycles such that $V(\sigma, \zeta) = V(\tau, \zeta) \neq \{0\}$. Then $\sigma = \tau$.*

Proof. We show that the map $\sigma \mapsto V(\sigma, \zeta)$ is a bijection when restricted to the ℓ-cycles σ such that $V(\sigma, \zeta) \neq 0$ by constructing its inverse. Fix such a cycle σ and label its entries as in (14) so that its image $V(\sigma, \zeta)$ is defined by equations (7) and (8) of Lemma 4.1. Working backwards, first note that $V(\sigma, \zeta)$ is a line, and therefore uniquely determines (7) and (8). Next observe that $\boldsymbol{\mu}_{m \vee d}$ is the disjoint union of the cosets $\boldsymbol{\mu}_m, \zeta\boldsymbol{\mu}_m, \ldots, \zeta^\ell\boldsymbol{\mu}_m$, so that for each i there is a *unique* scalar $\zeta^j(\delta_1\delta_2\cdots\delta_j)^{-1}$ appearing in (7) that is contained in the coset $\zeta^i\boldsymbol{\mu}_m$, from which one recovers $\delta_1\delta_2\cdots\delta_{i-1}$ and σ_i. The cycle σ is obtained by letting i range from 1 to ℓ while taking successive quotients so as to isolate each δ_i. $\qquad\square$

The general case follows:

Proposition 5.2. *Let ζ and ℓ be as in Lemma 5.1. Suppose that*

$$S := \{\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(q)}\} \quad and \quad T := \{\tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(q)}\}$$

are two sets of pairwise disjoint ℓ-cycles in $G(m, 1, n)$ that satisfy
(1) $\dim V(\sigma^{(i)}, \zeta) = \dim V(\tau^{(i)}, \zeta) = 1$ for all $i \in [q]$, and
(2) $\bigoplus_i V(\sigma^{(i)}, \zeta) = \bigoplus_i V(\tau^{(i)}, \zeta)$.
Then $S = T$.

Proof. Consider the image of $\bigoplus_i V(\sigma^{(i)}, \zeta)$ under the orthogonal projection of $\mathbb{C}^n$ onto $\bigoplus_{t \in \mathrm{Supp}(\tau^{(1)})} \mathbb{C}e_t$ and employ Lemma 4.1 to show that $V(\tau^{(1)}, \zeta)$ is equal to $V(\sigma^{(k)}, \zeta)$ for some k, then apply Lemma 5.1. The result follows by induction. $\quad\square$

We now come to the crux of this section.

Proposition 5.3. *Let ζ and ℓ be as in Lemma 5.1, and set $W = G(m, p, n)$. Let $E \in E(W, \zeta)$ be maximal under inclusion. Then there exists a set of pairwise disjoint ℓ-cycles $\{\sigma^{(1)}, \ldots, \sigma^{(a(d))}\} \subset G(m, 1, n)$ such that*

$$E = V(\sigma^{(1)}, \zeta) \oplus \ldots \oplus V(\sigma^{(a(d))}, \zeta).$$

Proof. Choose $g \in W$ such that $E = V(g, \zeta)$, and write $g = \tau^{(1)}\tau^{(2)}\cdots\tau^{(q)}$ as a product of disjoint cycles, indexed so that

$$\dim V(\tau^{(i)}, \zeta) = \begin{cases} 1 & \text{if } 1 \leq i \leq a(d); \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.1 tells us that $\ell(\tau^{(i)}) \geq \ell$ for each $i \in [a(d)]$, and so $n \geq a(d)\ell$. If $n = a(d)\ell$, the claim follows. Assume otherwise so that $n > a(d)\ell$, and set

$$h = \prod_{i=0}^{a(d)-1} \begin{pmatrix} i\ell + 1 & i\ell + 2 & \ldots & i\ell + \ell \\ \zeta^\ell(i\ell + 2) & i\ell + 3 & \ldots & i\ell + 1 \end{pmatrix}.$$

Let π' be the element of $G(m, 1, n)$ mapping e_n to $\zeta^{-a(d)\ell}e_n$ while fixing all other e_i. Clearly $h\pi' \in G(m, p, n)$ and

$$V(h\pi', \zeta) = V(\pi^{(0)}, \zeta) \oplus V(\pi^{(1)}, \zeta) \oplus \ldots \oplus V(\pi^{(a(d)-1)}, \zeta) \oplus V(\pi', \zeta)$$

for

$$\pi^{(i)} := \begin{pmatrix} i\ell+1 & i\ell+2 & \ldots & i\ell+\ell \\ \zeta^\ell(i\ell+2) & i\ell+3 & \ldots & i\ell+1 \end{pmatrix}.$$

Since Lemma 4.1 implies $\dim V(\pi^{(i)}, \zeta) = 1$ for $0 \le i \le a(d) - 1$, it follows that $V(h\pi', \zeta)$ is a maximal eigenspace in $E(W, \zeta)$. Hence the result, as W acts transitively on its maximal ζ-eigenspaces by Proposition 3.7(ii). $\square$

We can now label each maximal eigenspace of $G(m, p, n)$ by the cycles that it determines. In the next section we shall use this labeling of the maximal eigenspaces to construct a CL-shelling of $\widehat{E(W, \zeta)}$.

Corollary 5.4. *Let ζ and ℓ be as in Lemma 5.1, and set $W = G(m, p, n)$. Let $E \in E(W, \zeta)$ be a maximal eigenspace under inclusion. Then there exists a unique sequence*

$$\mathrm{word}(E) := (s_1, s_2, \ldots, s_{n-a(d)\ell} \; ; \; \sigma^{(1)}, \ldots, \sigma^{(a(d))})$$

of integers s_i and disjoint ℓ-cycles $\sigma^{(i)} \in G(m, 1, n)$ with the following properties:

(i) $E = \bigoplus_i V(\sigma^{(i)}, \zeta)$.
(ii) $\{s_1, s_2, \ldots, s_{n-a(d)\ell}\} = [n] \setminus \bigcup_{i=1}^{a(d)} \mathrm{Supp}(\sigma^{(i)})$.
(iii) $s_1 < s_2 < \ldots < s_{n-a(d)\ell}$.
(iv) $\min \mathrm{Supp}(\sigma^{(i)}) < \min \mathrm{Supp}(\sigma^{(j)})$ whenever $i < j$.

Definition 5.5. Linearly order $\boldsymbol{\mu}_m$ as follows: set $\delta := e^{2\pi i/m}$ and define $\delta^j < \delta^k$ whenever $0 \le j < k < m$. Suppose that σ and τ are two distinct cycles of the same length in $G(m, 1, n)$, and let k index the first column in which they differ:

$$\sigma = \begin{pmatrix} \sigma_1 & \sigma_2 & \ldots & \sigma_k & \ldots & \sigma_\ell \\ \delta_1\sigma_2 & \delta_2\sigma_3 & \ldots & \delta_k\sigma_{k+1} & \ldots & \delta_\ell\sigma_1 \end{pmatrix}$$

$$\tau = \begin{pmatrix} \tau_1 & \tau_2 & \ldots & \tau_k & \ldots & \tau_\ell \\ \epsilon_1\tau_2 & \epsilon_2\tau_3 & \ldots & \epsilon_k\tau_{k+1} & \ldots & \epsilon_\ell\tau_1 \end{pmatrix}.$$

Define $\sigma <_{\mathrm{lex}} \tau$ if the kth column of σ is lexicographically less than that of τ in the sense that one of the following holds.

(a) $\sigma_k < \tau_k$.
(b) $\sigma_k = \tau_k$ and $\sigma_{k+1} < \tau_{k+1}$.
(c) $\sigma_k = \tau_k$, $\sigma_{k+1} = \tau_{k+1}$, and $\delta_k < \epsilon_k$.

Definition 5.6. Let $W = G(m, p, n)$ and let ζ be a primitive d^{th} root of unity for $d > 1$. For two words

$$\mathrm{word}(E) = (s_1, \ldots, s_{n-a(d)\ell} \; ; \; \sigma^{(1)}, \ldots, \sigma^{(a(d))})$$
$$\mathrm{word}(E') = (t_1, \ldots, t_{n-a(d)\ell} \; ; \; \tau^{(1)}, \ldots, \tau^{(a(d))})$$

of distinct maximal eigenspaces $E, E' \in E(W, \zeta)$, define $\mathrm{word}(E) <_{\mathrm{lex}} \mathrm{word}(E')$ if in the first position in which they differ, the term of $\mathrm{word}(E)$ is strictly less than the corresponding term of $\mathrm{word}(E')$.

Example 5.7. Ordering the three maximal (-1)-eigenspaces E_i of $\mathfrak{S}_4$ by their words $\mathrm{word}(E_i)$, we have $E_1 <_{\mathrm{lex}} E_2 <_{\mathrm{lex}} E_3$ for E_i the eigenspace labeled by $\textcircled{i}$ in Figure 2; see Table 1. (Note that $n = a(d)\ell$ in this case.)

i	$E_i = \{\mathbf{z} \in \mathbb{C} \text{ satisfying } \ldots\}$	$\mathrm{word}(E_i)$
1	$z_1 = -z_2,\ z_3 = -z_4$	$\left(\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \right)$
2	$z_1 = -z_2,\ z_3 = -z_4$	$\left(\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \right)$
3	$z_1 = -z_2,\ z_3 = -z_4$	$\left(\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \right)$

TABLE 1. The maximal (-1)-eigenspaces E_i of $\mathfrak{S}_4$ in Figure 2, indexed with respect to lexicographic order on their words.

The maximal spaces in Figures 1–5 are similarly indexed. In the poset $E(\mathfrak{S}_4, \zeta)$ of Figure 3, for example, the eigenspaces

$$E_3 = \{\mathbf{z} \in \mathbb{C} \ : \ z_2 = 0, \ z_1 = \zeta z_3 = \zeta^2 z_4\}$$
$$E_5 = \{\mathbf{z} \in \mathbb{C} \ : \ z_3 = 0, \ z_1 = \zeta z_2 = \zeta^2 z_4\}$$

have words

$$\mathrm{word}(E_3) = \left(2, \begin{pmatrix} 1 & 3 & 4 \\ 3 & 4 & 1 \end{pmatrix} \right)$$
$$\mathrm{word}(E_5) = \left(3, \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{pmatrix} \right)$$

such that $\mathrm{word}(E_3) <_{\mathrm{lex}} \mathrm{word}(E_5)$.

6. $G(m, p, n)$ CASE OF THEOREM 1.1

For P a finite graded poset, denote its *rank function* by $r(x) : P \to \mathbb{Z}$ (with minimal elements having rank 0), and its *rank* by $r(P) := \max\{r(x) \ : \ x \in P\}$. Recall that a poset is *bounded* if it contains both a bottom element $\hat{0}$ and a top element $\hat{1}$, and that an *atom* in a poset with a $\hat{0}$ is any element that covers $\hat{0}$.

Definition 6.1 (Björner–Wachs [5]). A bounded poset P is said to *admit a recursive atom ordering* if its rank $r(P)$ is 1, or if $r(P) > 1$ and there is an ordering of the atoms $a_1, \ldots, a_t$ that satisfies the following.

  (i) Each interval $[a_j, \hat{1}]$ admits a recursive atom ordering in which its atoms that are contained in $[a_i, \hat{1}]$ for some $i < j$ come first.
  (ii) If $i < j$ and $a_i, a_j < x$, then there exists a $k < j$ and an atom $\widetilde{x}$ of $[a_j, \hat{1}]$ for which $a_k < \widetilde{x} \le x$.

A well-known result of Björner and Wachs [5] states that any ordering of the atoms in a *totally semimodular poset* is a recursive atom ordering. In particular, any ordering of the atoms in a semimodular lattice is a recursive atom ordering, from which the next useful lemma follows immediately.

Lemma 6.2 (Lemma 3 in [33]). *If P is a bounded poset in which $[a, \hat{1}]$ is a semimodular lattice for every atom $a \in P$, then an atom ordering $a_1, \ldots, a_t$ is a recursive atom ordering if and only if it satisfies condition* (ii) *of Definition 6.1.*

Our goal is to give, when $W = G(m,p,n)$ and ζ is a root of unity, a recursive atom ordering for $P = \widehat{E(W,\zeta)}$, whose atoms are the maximal ζ-eigenspaces of W under inclusion. By Lemma 6.2, this amounts to producing a candidate ordering and verifying condition (ii) of Definition 6.1. The case $d = 1$ is *trivial* in the sense that there is only one atom. For $d > 1$ we order the atoms by their words:

Theorem 6.3. *Let $W = G(m,p,n)$ and let ζ be a primitive d^{th} root of unity for $d > 1$. Then the lexicographic ordering of atoms E of $\widehat{E(W,\zeta)}$ by their words* word(E) *is a recursive atom ordering.*

Proof. By Theorem 1.2 and Lemma 6.2, we need only verify that the atom ordering satisfies condition (ii) of Definition 6.1. The result follows immediately if there is only one atom, so assume otherwise and note that $a(d) < n$ by Corollary 3.8.

Suppose that $A, B \in E(W,\zeta)$ are two atoms with word$(A) <_{\text{lex}}$ word(B). Write

$$\text{word}(A) = (a_1, \ldots, a_{n-a(d)\ell} \; ; \; \sigma^{(1)}, \ldots, \sigma^{(a(d))})$$
$$\text{word}(B) = (b_1, \ldots, b_{n-a(d)\ell} \; ; \; \tau^{(1)}, \ldots, \tau^{(a(d))}).$$

Since any element that lies above both A and B in $E(W,\zeta)$ must be a subspace of $A \cap B$, it suffices to exhibit a maximal eigenspace $C \in E(W,\zeta)$ that satisfies

(i) word$(C) <_{\text{lex}}$ word(B), and
(ii) $A \cap B \subseteq B \cap H \subseteq C$ for some hyperplane $H \in \mathscr{L}_{W'}$,

where W' is as in Proposition 4.3.

Choose (possibly empty) cycles $\sigma^{(0)}, \tau^{(0)} \in G(m,1,n)$ such that

$$g := \sigma^{(0)}\sigma^{(1)} \cdots \sigma^{(a(d))}$$
$$h := \tau^{(0)}\tau^{(1)} \cdots \tau^{(a(d))}$$

are products of disjoint cycles with $g, h \in W$. Then $A = V(g,\zeta)$ and $B = V(h,\zeta)$, since $A = \bigoplus_{i \geq 1} V(\sigma^{(i)}, \zeta)$ and $B = \bigoplus_{i \geq 1} V(\tau^{(i)}, \zeta)$ are maximal.

Case 1. $a_i \neq b_i$ for some i.

Let i be the smallest such index. Then $a_i < b_i$ and $a_i \in \text{Supp}(\tau^{(j)})$ for some $j \geq 1$. Set $C := rB = V(rhr^{-1}, \zeta)$ for

$$r = \begin{pmatrix} a_i & b_i \\ b_i & a_i \end{pmatrix}.$$

Since one obtains word(C) by interchanging a_i and b_i in word(B), it follows that word$(C) <_{\text{lex}}$ word(B). Applying r to both sides of $B \cap H_r \subseteq B$ shows that $B \cap H_r \subseteq C$. Lastly, $A \cap B \subseteq B \cap H_r$ follows from the fact that $z_{a_i} = 0$ on A and $z_{b_i} = 0$ on B.

For example, if $W = G(4,2,8)$ and $\zeta = e^{2\pi i/6}$, then for $\omega := e^{2\pi i/4}$ and

$$\text{word}(A) = \left(4, 7, \begin{pmatrix} 1 & 3 & 2 \\ \omega 3 & -2 & \omega 1 \end{pmatrix}, \begin{pmatrix} 5 & 6 & 8 \\ -6 & -8 & 5 \end{pmatrix}\right)$$
$$\text{word}(B) = \left(4, 8, \begin{pmatrix} 1 & 6 & 5 \\ -6 & 5 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 7 & 3 \\ 7 & 3 & 2 \end{pmatrix}\right),$$

one has $r = \begin{pmatrix} 7 & 8 \\ 8 & 7 \end{pmatrix}$ and

$$\text{word}(C) = \left(4, 7, \begin{pmatrix} 1 & 6 & 5 \\ -6 & 5 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 8 & 3 \\ 8 & 3 & 2 \end{pmatrix} \right).$$

Case 2. $a_i = b_i$ for all i.

Let $j \geq 1$ be the smallest integer for which $\sigma^{(j)} \neq \tau^{(j)}$. Set $\sigma := \sigma^{(j)}$ and $\tau := \tau^{(j)}$, and let k index the first column in which σ and τ differ:

$$\sigma = \begin{pmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_k & \cdots & \sigma_\ell \\ \delta_1 \sigma_2 & \delta_2 \sigma_3 & \cdots & \delta_k \sigma_{k+1} & \cdots & \delta_\ell \sigma_1 \end{pmatrix}$$

$$\tau = \begin{pmatrix} \tau_1 & \tau_2 & \cdots & \tau_k & \cdots & \tau_\ell \\ \epsilon_1 \tau_2 & \epsilon_2 \tau_3 & \cdots & \epsilon_k \tau_{k+1} & \cdots & \epsilon_\ell \tau_1 \end{pmatrix}.$$

Note that $\sigma_i = \tau_i$ for $i \leq k$, and $\delta_i = \epsilon_i$ for $i < k$.

By Lemma 4.1, each $\mathbf{z} \in A \cap B$ satisfies the two equations

$$z_{\sigma_1} = \zeta^k \delta_1^{-1} \cdots \delta_k^{-1} z_{\sigma_{k+1}}$$

$$z_{\tau_1} = \zeta^k \epsilon_1^{-1} \cdots \epsilon_k^{-1} z_{\tau_{k+1}}.$$

Since $\sigma_1 = \tau_1$ and $\delta_i = \epsilon_i$ for $i < k$, it follows that

(15) $$z_{\sigma_{k+1}} = \delta_k \epsilon_k^{-1} z_{\tau_{k+1}} \quad \text{for all} \quad \mathbf{z} \in A \cap B.$$

Case 2a. $\sigma_{k+1} < \tau_{k+1}$.

Equation (15) says that $A \cap B \subseteq H_r$ for the reflection

$$r = \begin{pmatrix} \sigma_{k+1} & \tau_{k+1} \\ \epsilon_k \delta_k^{-1} \tau_{k+1} & \delta_k \epsilon_k^{-1} \sigma_{k+1} \end{pmatrix}.$$

Set $C := rB = V(rhr^{-1}, \zeta)$ so that, as in Case 1, one has $A \cap B \subseteq B \cap H_r \subseteq C$. Then $\text{word}(C) <_{\text{lex}} \text{word}(B)$, since σ_{k+1} occurs to the right of τ_{k+1} in $\text{word}(B)$. (More precisely, either $\sigma_{k+1} = \tau_{k+1+i}$ for some $i \geq 1$, or σ_{k+1} is in the support of $\tau^{(l)}$ for some $l > j$.)

For example, if $W = G(4, 2, 9)$ and $\zeta = e^{2\pi i/16}$, then for $\omega := e^{2\pi i/4}$ and

$$\text{word}(A) = \left(4, \begin{pmatrix} 1 & 3 & 2 & 7 \\ \omega 3 & -2 & -7 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 9 & 6 & 8 \\ 9 & \omega 6 & 8 & 5 \end{pmatrix} \right)$$

$$\text{word}(B) = \left(4, \begin{pmatrix} 1 & 3 & 9 & 8 \\ \omega 3 & \omega^{-1} 9 & 8 & \omega 1 \end{pmatrix}, \begin{pmatrix} 2 & 6 & 5 & 7 \\ \omega^{-1} 6 & 5 & -7 & 2 \end{pmatrix} \right),$$

one has $k = 2$, $r = \begin{pmatrix} 2 & 9 \\ \omega 9 & \omega^{-1} 2 \end{pmatrix}$, and

$$\text{word}(C) = \left(4, \begin{pmatrix} 1 & 3 & 2 & 8 \\ \omega 3 & -2 & \omega 8 & \omega 1 \end{pmatrix}, \begin{pmatrix} 5 & 7 & 9 & 6 \\ -7 & \omega 9 & -6 & 5 \end{pmatrix} \right).$$

Case 2b. $\sigma_{k+1} = \tau_{k+1}$.

Then $\delta_k < \epsilon_k$. It follows from (15) that $z_{\tau_{k+1}} = 0$, and hence $z_i = 0$ whenever $i \in \text{Supp}(\tau)$, for each $\mathbf{z} \in A \cap B$. In particular, $A \cap B \subseteq B \cap H_r = B \cap H_s$ for

$$r = \begin{pmatrix} \tau_k \\ \delta_k \epsilon_k^{-1} \tau_k \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} \tau_\ell \\ \delta_k^{-1} \epsilon_k \tau_\ell \end{pmatrix}.$$

Set $h' := hrs \in W$ and $C := V(h', \zeta)$. First note that C is maximal, since B is maximal and the cycle τrs in h' has $\dim V(\tau rs, \zeta) = 1$ by Lemma 4.1. It is also clear that $B \cap H_r = C \cap H_r$, and hence $A \cap B \subseteq B \cap H_r \subseteq C$. It remains to see that $C \neq B$, from which it follows that $\mathrm{word}(C) <_{\mathrm{lex}} \mathrm{word}(B)$. To this end, observe that $k < \ell$, since since $\delta_i = \epsilon_i$ for $i < k$ and $\delta_1 \cdots \delta_\ell = \zeta^\ell = \epsilon_1 \cdots \epsilon_\ell$, while $\delta_k < \epsilon_k$. It follows that $\tau rs \neq \tau$, and therefore $C \neq B$ by Proposition 5.2.

For example, if $W = G(4, 2, 8)$ and $\zeta = e^{2\pi i/6}$, then for $\omega := e^{2\pi i/4}$ and

$$\mathrm{word}(A) = \left(4, 7, \begin{pmatrix} 1 & 3 & 2 \\ \omega 3 & -2 & \omega 1 \end{pmatrix}, \begin{pmatrix} 5 & 6 & 8 \\ -6 & -8 & 5 \end{pmatrix}\right)$$

$$\mathrm{word}(B) = \left(4, 7, \begin{pmatrix} 1 & 3 & 2 \\ -3 & 2 & -1 \end{pmatrix}, \begin{pmatrix} 5 & 8 & 6 \\ 8 & 6 & 5 \end{pmatrix}\right),$$

one has $k = 1$, $r = \begin{pmatrix} 1 \\ \omega^{-1}1 \end{pmatrix}$, $s = \begin{pmatrix} 2 \\ \omega 2 \end{pmatrix}$, and

$$\mathrm{word}(C) = \left(4, 7, \begin{pmatrix} 1 & 3 & 2 \\ \omega 3 & 2 & \omega^{-1}1 \end{pmatrix}, \begin{pmatrix} 5 & 8 & 6 \\ 8 & 6 & 5 \end{pmatrix}\right). \qquad \square$$

7. Exceptional cases of Theorems 1.1 and 1.2

Here we establish CL-shellability for exceptional reflection groups, excluding for types E_6, E_7, E_8. This will conclude our proof of Theorem 1.1. We also finish proving Theorem 1.2 by establishing (1) for all exceptional reflection groups, with no exclusions. Central to all of this is Theorem 7.1 below, which pins down the different posets $E(W, \zeta)$ by characterizing when two eigenvalues give the same poset. Recall that for a reflection group W and positive integer d we write $A(d)$ for the multiset of degrees that are divisible by d, so that in particular $a(d) = |A(d)|$ is the dimension of any maximal ζ-eigenspace of the group.

Theorem 7.1. *Let W be a reflection group, and let ζ and ζ' be roots of unity of orders d and d', respectively. Then $E(W, \zeta) = E(W, \zeta')$ if and only if $A(d) = A(d')$.*

We are most interested in the case when $E(W, \zeta)$ has rank at least two and is not $\mathscr{L}_W$, so that $a(d) \neq 0, 1, n$, where n is the rank of W. This is when CL-shellability and (1) are nontrivial. Table 2 lists all instances of an exceptional reflection group and positive integer d such that $a(d) \neq 0, 1, n$, so for each d we are interested in $E(W, \zeta)$ for each primitive d^{th} root of unity ζ. The d's are grouped according to $A(d)$, so Theorem 7.1 asserts that $E(W, \zeta) = E(W, \zeta')$ if and only if d, d' appear together in Table 2. In other words, Theorem 7.1 reduces the nontrivial exceptional cases of Question A and (1) to just 23 different posets $E(W, \zeta)$, one for each set of d's in Table 2.

Before proving Theorem 7.1, we use general methods to establish CL-shellability and (1) when d is a *regular number* satisfying $a(d) = 2$. These are Corollaries 7.6 and 7.4 below, and with the help of one of them we then prove Theorem 7.1. Table 2 shows that 14 of its 23 cases satisfy the hypotheses of Corollaries 7.6 and 7.4. For the other 9 cases (Table 3 below) we first outline our basic computer verification of (1) for all of the cases, with no exclusions, then we outline our basic computer search to find a recursive atom ordering (and thus a CL-shelling) for 5 of the 9 cases. We end by listing the 4 open cases of the strong and weak versions of Question A involving types E_6, E_7, E_8 and various roots of unity, which we write down in Table 4 below.

W	$a(d)$	d	W	$a(d)$	d	W	$a(d)$	d
G_{25}	2	2,6	G_{32}	2	4,12	G_{36}	2*	4
G_{28}	2	3,6	G_{33}	2*	4		3	3,6
	2	4		3	3,6	G_{37}	2	5,10
G_{30}	2	3,6	G_{34}	2*	4,12		2	8
	2	4	G_{35}	2	4		2	12
	2	5,10		2	6		4	3,6
G_{31}	2	3,6,12		3	3		4	4
	2	8		4	2			

TABLE 2. All instances of an exceptional reflection group W of rank n and positive integer d such that $a(d) \neq 0, 1, n$. Nonregular cases are indicated by *, and values d, d' appear together if and only if $A(d) = A(d')$; see Theorem 7.1. See also Table 10 below.

We need some facts from Springer's theory of regular elements [35], where an element g of a (finite) reflection group $W \subset \mathrm{GL}(V)$ is called ζ-*regular* if it has a ζ-eigenvector v that is not contained in any reflecting hyperplane for W. When such an element g exists, the eigenvalue ζ is called a *regular eigenvalue*, and the order d of ζ is a *regular number*. For such a number d, Springer [35] showed that an element $h \in W$ is ζ-regular if and only if $\dim V(h, \zeta) = a(d)$, and by a result of Springer and Lehrer [24], these regular numbers d of W are easy to compute:

Theorem 7.2 (Lehrer–Springer). *For any complex reflection group, a positive integer d is a regular number if and only if it divides as many degrees as it does codegrees.*

Lemma 7.3. *Let W be a reflection group and let ζ be a primitive d^{th} root of unity. Suppose that d is regular and that $E = V(g, \zeta)$ is a maximal ζ-eigenspace of W under inclusion. Then for any reflection $r \in W$ one has $E \cap H_r = V(gr, \zeta)$, where $H_r := \ker(1 - r)$ denotes the reflecting hyperplane of r.*

Proof. Suppose that the trivial inclusion $E \cap H_r \subseteq V(gr, \zeta)$ is proper. Then by considering dimension, $V(gr, \zeta)$ is necessarily maximal. It follows from a standard argument using Proposition 3.7(ii) and a theorem of Steinberg [39, Thm. 1.5] that gr is therefore conjugate to g. In particular, $\det(gr) = \det(g)$. But $\det(r) \neq 1$. $\square$

Applying Proposition 3.10(ii) to the case when $\dim E = 2$ gives the following.

Corollary 7.4. *Maintain the notation and assumptions of Lemma 7.3, and suppose in addition that $a(d) = 2$. Then $E(W, \zeta)$ satisfies* (1).

Another consequence of Lemma 7.3 is that $E(W, \zeta)$ is connected.

Corollary 7.5. *Maintain the notation and assumptions of Lemma 7.3, and suppose in addition that $a(d) \geq 2$. Then (the Hasse diagram of) the poset $E(W, \zeta) \setminus \{\hat{1}\}$ is connected (as a graph). Equivalently, $\Delta(E(W, \zeta) \setminus \{\hat{1}\})$ is connected.*

Proof. Consider two maximal eigenspaces $E, E' \in E(W, \zeta)$ and choose an element $g \in W$ such that $E' = gE$ (possible by Proposition 3.7(ii)). Write g as a product of reflections $r_k r_{k-1} \cdots r_1$ and define $E_i = r_i r_{i-1} \cdots r_1 E$ for each i so that

$$E = E_0, \ E_1, \ E_2, \ \ldots, \ E_k = E'$$

is a sequence of maximal eigenspaces. Since the intersection of a neighboring pair E_i, E_{i+1} is $H_{r_{i+1}} \cap E_i$, an element of $E(W, \zeta)$ by Lemma 7.3, we conclude that E is connected to E' in the Hasse diagram of $E(W, \zeta)$, and hence the result. $\square$

Corollary 7.6. *Maintain the notation and assumptions of Lemma 7.3, and suppose in addition that $a(d) = 2$. Then $\widehat{E(W, \zeta)}$ is CL-shellable.*

Proof of Theorem 7.1. By Proposition 3.1, we may assume that W is irreducible.

Assume that $A(d) \neq A(d')$. It is well known [35, Proof of Thm. 3.4(i)] that for any $S \subset [n]$, each irreducible component of $\bigcap_{i \in S} H_i$ has dimension $n - |S|$, where recall from Proposition 3.5 that H_i is the hypersurface defined by the invariant polynomial f_i of a set of basic invariants $f_1, f_2, \ldots, f_n$ whose respective degrees are $d_1, d_2, \ldots, d_n$. In particular,

$$\text{(16)} \qquad \bigcap_{d \nmid d_i} H_i \neq \left(\bigcap_{d \nmid d_i} H_i \right) \cap \left(\bigcap_{d' \nmid d_i} H_i \right),$$

for indeed, $n - |A(d)|$ is not equal to $n - |A(d) \cup A(d')|$. It follows from (16) that $\bigcap_{d \nmid d_i} H_i \neq \bigcap_{d' \nmid d_i} H_i$, and so $E(W, \zeta) \neq E(W, \zeta')$ by Proposition 3.5.

Assume that $A(d) = A(d')$ so that $a(d) = a(d')$, and consider the following cases.
Case 1. Either $W = G(m, p, n)$ or $a(d) \in \{0, 1, n\}$. Theorem 1.2 is certainly true when $a(d) \in \{0, 1, n\}$, and was established in §4 for $G(m, p, n)$. We thus have

$$\text{(17)} \qquad E(W, \zeta) = \{E \cap X \ : \ X \in \mathscr{L}_W \text{ and } E \in E(W, \zeta) \text{ maximal}\},$$

$$\text{(18)} \qquad E(W, \zeta') = \{E \cap X \ : \ X \in \mathscr{L}_W \text{ and } E \in E(W, \zeta') \text{ maximal}\}.$$

Now observe that the right sides of (17) and (18) agree by Corollary 3.6.
Case 2. W exceptional, $a(d) = 2$, and d regular. In this case (17) and (18) follow from Corollary 7.4, and so again $E(W, \zeta) = E(W, \zeta')$ by Corollary 3.6.
Case 3. W exceptional, and either $3 \leq a(d) \leq n - 1$ or d is not regular. It clearly suffices to assume that d is the smallest (positive) value for which $A(d) = A(d')$, so that ζ and ζ^{-1} are the only primitive d^{th} roots of unity by Table 2. If $d = d'$, then the equality $E(W, \zeta) = E(W, \zeta')$ follows from the fact that $V(g, \zeta) = V(g^{-1}, \zeta^{-1})$. If $d < d'$, the classification shows that there exists a prime p such that $\boldsymbol{\mu}_{d'} = \boldsymbol{\mu}_p \boldsymbol{\mu}_d$ and $a(p) = n$. Choose ζ_p of order p and ζ_d of order d such that $\zeta' = \zeta_p \zeta_d$ and note that $\zeta_p \in W$ by Proposition 3.8(ii). Since $V(g, \zeta_p \zeta_d) = V(\zeta_p^{-1} g, \zeta_d)$, it follows that $E(W, \zeta') = E(W, \zeta_d)$, while $E(W, \zeta_d) = E(W, \zeta)$ from the previous case $d = d'$. $\square$

W	ζ
G_{35}	-1
G_{33}, G_{35}, G_{36}, G_{37}	$e^{2\pi i/3}$
G_{33}, G_{34}, G_{36}, G_{37}	$e^{2\pi i/4}$

TABLE 3. Cases in Table 2 when $a(d) \geq 3$ or d is not a regular number.

In 14 of the 23 cases from Table 2, d is a regular number satisfying $a(d) = 2$, so for these we have CL-shellability and (1) by Corollaries 7.6 and 7.4. The remaining 9 cases of Question A (both versions) and of Theorem 1.2 are listed in Table 3.

For the 9 remaining cases of Theorem 1.2 listed in Table 3 we used computer algebra software MAGMA to verify (1) in each case by first choosing[2] an element

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[2]For example, g may be taken to be any *Coxeter element* raised to the power d_n/d in each of the regular cases except $W = G_{37}, \zeta = e^{2\pi i/4}$ when $d \nmid d_n$.

$g \in W$ whose eigenspace $E := V(g, \zeta)$ is maximal (i.e., of dimension $a(d)$ by Proposition 3.7(i)), then constructing and checking that the set $\{E \cap X \ : \ X \in \mathscr{L}_W\}$ is contained in the set

$$\{V(gr_1 \cdots r_k, \zeta) \ : \ r_i \text{ is a reflection of } W \text{ and } 0 \le k \le a(d)\},$$

from which the desired equality (1) follows by Proposition 3.10(ii).

Regarding the 9 remaining cases of Question A, we first constructed the upper interval $[E, \hat{1}]$ (with g and E as above), then from the transitivity of W on its maximal ζ-eigenspaces, we constructed all such maximal intervals of $E(W, \zeta)$ by taking the $W/N_W(E)$-orbit of $[E, \hat{1}]$. When g is regular, we used the additional fact that the normalizer $N_W(E)$ is the centralizer $Z_W(g) := \{h \in W \ : \ hg = gh\}$ of g in W. From the collection of these upper intervals, in the three nonregular cases $(G_{33}, G_{34}, G_{36}$ with $\zeta = e^{2\pi i/4}$ so that $a(d) = 2)$ we easily verified that $E(W, \zeta)$ is connected, so that $\widehat{E(W, \zeta)}$ is CL-shellable. For the single remaining case of Theorem 1.1 (when $W = G_{33}$ and $\zeta = e^{2\pi i/3}$) we chose at random one hundred orderings $E = E_0, E_1, \ldots, E_{40}$ of the maximal eigenspaces (atoms) of $E(W, \zeta)$ with the property that

$$\mathrm{dist}(E, E_1) \le \mathrm{dist}(E, E_2) \le \mathrm{dist}(E, E_3) \le \ldots \le \mathrm{dist}(E, E_{40}),$$

where $\mathrm{dist}(E, E_i)$ denotes the graph-theoretic distance from E to E_i in the restriction of $E(W, \zeta)$ to its bottom two ranks, i.e., to those eigenspaces of dimension either $a(d)$ or $a(d) - 1$. We found that each of these orderings was, in fact, a recursive atom ordering. Lastly, we remark that we also found (an unenlightening) recursive atom ordering for $E(G_{35}, -1)$.

W	ζ	
G_{35} (type E_6)	$e^{2\pi i/3}$	
G_{36} (type E_7)	$e^{2\pi i/3}$	
G_{37} (type E_8)	$e^{2\pi i/3}$	$e^{2\pi i/4}$

TABLE 4. Open cases of the strong and weak versions of Question A.

The remaining 4 open cases of the strong and weak versions of Question A are listed in Table 4. For more evidence, we used MAGMA to show that $E(W, \zeta)$ is Cohen–Macaulay when $W = G_{35}$ and $\zeta = e^{2\pi i/3}$, and we are hopeful that similar computations can be carried out for the remaining groups.

8. CONSEQUENCES FOR THE TOPOLOGY OF EIGENSPACE ARRANGEMENTS

Recall that the order complex of a finite poset P is the (abstract) simplicial complex $\Delta(P)$ consisting of all totally ordered sets $x_1 < x_2 < \ldots < x_i$ in P. We adopt the convention of writing $\widetilde{H}_i(P)$ for its ith reduced homology group $\widetilde{H}_i(\Delta(P), \mathbb{C})$, and $\widetilde{H}_i(x, y)$ for $\widetilde{H}_i(\Delta((x, y)), \mathbb{C})$, where (x, y) denotes the open interval formed by $x, y \in P$. If P is a G-poset, then $\Delta(P)$ inherits an action of G from P, and thus each $\widetilde{H}_i(P)$ may be regarded as a G-module (i.e., a $\mathbb{C}[G]$-module) by functoriality. One may alternatively consider the module afforded by the ith reduced cohomology group $\widetilde{H}^i(P) = \widetilde{H}^i(\Delta(P), \mathbb{C})$, but this representation is just dual to $\widetilde{H}_i(P)$.

Remark 8.1. We adopt the following conventions for the rank and order complex of an empty open interval $I = (x, y)$ and the empty poset $P \setminus \{\hat{1}\}$ when $|P| = 1$. If y covers x, we set $r(I) = r(P \setminus \{\hat{1}\}) = -1$ and take $\Delta(x, y)$ and $\Delta(P \setminus \{\hat{1}\})$ to be the (-1)-dimensional complex $\{\varnothing\}$ containing only the empty face; if $x = y$, we set $r(I) = -2$ and regard $\Delta(x, x)$ as a (-2)-dimensional *degenerate empty complex* $\varnothing$ with no faces at all. Then $\widetilde{H}_i(\{\varnothing\})$ is $\mathbb{C}$ when $i = -1$, and 0 otherwise, while we let the reduced homology of $\varnothing$ vanish in all dimensions and define $\widetilde{H}_{-2}(\varnothing) := \mathbb{C}$.

For our purposes, call a finite collection $\mathcal{A}$ of proper complex linear subspaces of $\mathbb{C}^n$ an *arrangement*. Associate with $\mathcal{A}$ its *intersection lattice* $L(\mathcal{A})$, obtained by ordering all intersections of subspaces in $\mathcal{A}$ by reverse inclusion. Note that the top element $\hat{1}$ of $L(\mathcal{A})$ is the (nonempty) intersection over $\mathcal{A}$, and that the bottom element $\hat{0}$ is $\mathbb{C}^n$, the intersection over the empty set. The following celebrated result of Goresky and MacPherson [15] says that the cohomology of the complement $\mathcal{M}_{\mathcal{A}} := \mathbb{C}^n \setminus \bigcup_{X \in \mathcal{A}} X$ is determined by the combinatorial data of the arrangement.

Theorem 8.2 (Goresky–MacPherson)**.** *Let $\mathcal{A}$ be a complex arrangement. Then*

$$\widetilde{H}^i(\mathcal{M}_{\mathcal{A}}) = \bigoplus_{x \in L(\mathcal{A}) \setminus \{\hat{0}\}} \widetilde{H}_{2 \operatorname{codim} x - i - 2}(\hat{0}, x) \quad \text{for all } i.$$

Call an arrangement $\mathcal{A}$ a *G-arrangement* if $G \subset \mathrm{GL}(\mathbb{C}^n)$ is such that $gX \in \mathcal{A}$ for every $X \in \mathcal{A}$ and $g \in G$. For such an arrangement both the cohomology of $\mathcal{M}_{\mathcal{A}}$ and the homology of $L(\mathcal{A})$ inherit a G-module structure, and Sundaram and Welker [41] established the following equivariant formulation of Theorem 8.2.

Theorem 8.3 (Sundaram–Welker)**.** *For a (finite, complex, linear) G-arrangement $\mathcal{A}$, one has a G-module isomorphism*

$$(19) \qquad \widetilde{H}^i(\mathcal{M}_{\mathcal{A}}) \cong \bigoplus_x \mathrm{Ind}_{G_x}^G \widetilde{H}_{2 \operatorname{codim} x - i - 2}(\hat{0}, x) \qquad \text{for all } i,$$

where x runs over a collection of representatives for the G-orbits on $L(\mathcal{A}) \setminus \{\hat{0}\}$, and G_x denotes the stabilizer $\{g \in G \ : \ gx = x\}$ for $x \in L(\mathcal{A})$.

A consequence of Theorem 1.1, Theorem 4.4, and Theorem 8.3 is that for $W = G(m, p, n)$, one has that the complement $\mathcal{M}(W, d)$ of the proper ζ-eigenspaces of W has torsion-free integral cohomology, and moreover, the top nonvanishing cohomology module of $\mathcal{M}(W, d)$ is equivariantly isomorphic to the top homology module of the proper part of the poset of ζ-eigenspaces:

Proposition 8.4. *Let W be an irreducible non-exceptional reflection group of rank n. That is, W is either $W(A_n)$ or $G(m, p, n)$ with $m > 1$ and $(m, p, n) \neq (2, 2, 2)$. Let ζ be a primitive d^{th} root of unity and let $\mathcal{A}$ be the arrangement of all proper ζ-eigenspaces of W, so that $\mathcal{M}_{\mathcal{A}} = \mathcal{M}(W, d)$. Then $L(\mathcal{A}) = E(W, \zeta) \cup \{\mathbb{C}^n\}$ and $\mathcal{M}(W, d)$ has torsion-free integral cohomology. Moreover, the following hold:*

(i) *If $a(d) = n$, so that $\mathbb{C}^n \in E(W, \zeta)$, then as W-modules*

$$\widetilde{H}^{2n-a(d)}(\mathcal{M}(W, d)) \ \cong \ \widetilde{H}_{a(d)-2}(\overline{E(W, \zeta)}).$$

(ii) *If $a(d) \neq n$, so that $\mathbb{C}^n \notin E(W, \zeta)$, then as W-modules*

$$\widetilde{H}^{2n-a(d)-1}(\mathcal{M}(W, d)) \ \cong \ \widetilde{H}_{a(d)-1}(\overline{E(W, \zeta)}).$$

Proof. The first claim, that $L(\mathcal{A}) = E(W, \zeta) \cup \{\mathbb{C}^n\}$, follows from the fact that $E(W, \zeta)$ is an upper order ideal of an intersection lattice $\mathscr{L}_{W'}$; see Theorem 4.4. Theorem 1.1 implies that all of the homology summands in Theorem 8.2 are free abelian, since shellability is inherited by open intervals. Hence the second claim. For (ii), suppose that $a(d) \neq n$, so that the intersection lattice $L(\mathcal{A})$ is obtained from $E(W, \zeta)$ by adjoining the new element $\mathbb{C}^n$. Because the top element of $L(\mathcal{A})$ has dimension 0 by irreducibility, it follows that the rank of an interval $(\hat{0}, x)$ is equal to $a(d) - \dim x - 1$ whenever $x \neq \hat{0}$. Taking i to be $2n - a(d) - 1$, the homology summands $\widetilde{H}_{2\operatorname{codim}x - i - 2}(\hat{0}, x)$ in (19) become $\widetilde{H}_{a(d) - 2\dim x - 1}(\hat{0}, x)$. Since the open intervals are shellable, such a summand vanishes if $a(d) - 2\dim x - 1$ is not equal to the dimension $a(d) - \dim x - 1$ of $\Delta(\hat{0}, x)$, and so the only surviving summand of (19) corresponds to the top element $x = \hat{1}$ of dimension 0. (i) is similar. $\square$

The first case of Proposition 8.4 is well known, since $E(W, 1) = \mathscr{L}_W$ is a geometric lattice. The homology module of $\Delta(\overline{\mathscr{L}_W})$ for $W = W(A_n)$ has received considerable attention; see Section 1. In this case Stanley [38] used work of Hanlon [18] to express the top homology module, and thus the top cohomology module of $\mathcal{M}(W(A_n), 1)$, as an induced linear representation. Lehrer and Solomon [22] extended Stanley's results to all cohomology groups of $\mathcal{M}(W(A_n), 1)$, and conjectured an extension to all (complexified) finite Coxeter groups. In the next section we will extend Stanley's result to arbitrary eigenvalues ζ. We have not explored the possibility of extending Lehrer and Solomon's result.

9. Combinatorics of $E(W, \zeta)$ in type A_{n-1}

Recall that $E(\mathfrak{S}_n, 1)$ is the well-understood intersection lattice for the braid arrangement. As such, it may be considered as the lattice of all set partitions of $\{1, 2, \ldots, n\}$ ordered by refinement. This simple combinatorial model extends to the intersection lattice for $G(r, 1, n)$ through a construction of Dowling [12], which in turn provides a model for $E(W, \zeta)$ via Theorem 4.4 when $W = G(m, p, n)$ and $E(W, \zeta) \neq \mathscr{L}_W$. Theorem 4.4 also tells us that the model inherited by $E(W, \zeta)$ is again particularly simple when the minimal elements have a simple description as elements of $\mathscr{L}_{W'}$, where $W' = G(m \vee d, 1, n)$ and d is the order of ζ. In this section we set $W = \mathfrak{S}_n$ and $d > 1$. Not only is the resulting combinatorial description for $E(\mathfrak{S}_n, \zeta)$ interesting in its own right, but it naturally distinguishes the poset of *d-divisible partitions* and leads to a precise description of the homotopy type of $E(\mathfrak{S}_n, \zeta) \setminus \{\hat{1}\}$ in terms of integer partitions. Throughout, $d > 1$ and ζ denotes a primitive d^{th} root of unity.

9.1. **Balanced Partitions.** The purpose of this subsection is to give a combinatorial description of $E(\mathfrak{S}_n, \zeta)$. As outlined above, we do so by viewing $E(\mathfrak{S}_n, \zeta)$ as an upper order ideal of $\mathscr{L}_W$ for $W = G(d, 1, n)$, regarded as a Dowling lattice. Though Dowling's original notation [12] is convenient for general groups, the lattice $\mathscr{L}_W$ is constructed as a Dowling lattice from the cyclic group $\mathbb{Z}/d\mathbb{Z}$ and lends itself to description in terms of integrally weighted set partitions, a notation that we shall use throughout.

Recall that a *set partition* of $\{0, 1, \ldots, n\}$ is a collection $\pi = \{B_0, B_1, \ldots, B_\ell\}$ of nonempty disjoint sets with $\bigcup_{i=0}^{\ell} B_i = \{0, 1, \ldots, n\}$. We call ℓ the *length* of π and write $\ell(\pi) := \ell$. Henceforth, we shall assume that the *blocks* are indexed so that B_0 is the *zero block*, i.e., that $0 \in B_0$. We also adopt the convention of writing such a

partition as $B_0/B_1/\ldots/B_\ell$, omitting commas and set braces for individual blocks when working with explicit sets of integers.

From a partition $B_0/B_1/\ldots/B_\ell$ of $\{0,1,\ldots,n\}$ and a positive integer m, one obtains an m-*weighted partition* by assigning a weight $w_i \in \mathbb{Z}/m\mathbb{Z}$ to each element $i \in B_1 \cup B_2 \cup \cdots \cup B_\ell$, while elements of B_0 remain unweighted. Indicating weights with superscripts, we have $015/2^0 3^4/4^0$ and $023/1^2 5^2/4^4$ as examples of weighted partitions for $n = 5$, with 023 being the zero block of the latter. We shall also find it useful to depict each weight by an equal number of bars, in which case we superfluously underline the zero block in order to emphasize that its elements bear no weights; see Figure 7 and 8.

For B a nonzero block of a weighted partition, let B^w denote the block obtained from B by adding w to the weight of each of its elements (modulo d), and let B^{erase} denote the block obtained by removing all weights. For example, if $B = 1^3 3^0 4^4$ and $d = 5$, then $B^3 = 1^1 3^3 4^2$ and $B^{\mathrm{erase}} = 134$.

Since $d > 1$, the arrangement of reflecting hyperplanes of $G(d,1,n)$ is given by

$$\mathcal{A}_{n,d} := \{z_i = 0 \;:\; 1 \le i \le n\} \cup \{z_i = \zeta^w z_j \;:\; 1 \le i < j \le n,\ w \in \mathbb{Z}/d\mathbb{Z}\}.$$

The intersection lattice $L(\mathcal{A}_{n,d})$ is $\mathfrak{S}_n$-equivariantly isomorphic (in fact, $G(d,1,n)$-isomorphic) to the (particular) Dowling lattice $\Pi_{n,d}$ defined as follows:

Definition 9.1. $\Pi_{n,d}$ is the poset of all $(d\text{-})$weighted partitions $\pi = B_0/\ldots/B_\ell$ of $\{0,\ldots,n\}$ with partial order defined by setting $\pi_1 \le \pi_2$ if for each block B of π_1 there exists a block of π_2 that either contains B^{erase} or contains B^w for some w; see Figure 7. It is the *Dowling lattice* constructed from the cyclic group $\mathbb{Z}/d\mathbb{Z}$; see [12].

We consider $\Pi_{n,d}$ as an $\mathfrak{S}_n$-poset by defining $\sigma(B_0/\ldots/B_\ell) := \sigma B_0/\ldots/\sigma B_\ell$, and the above-mentioned equivariant isomorphism $\Pi_{n,d} \xrightarrow{\sim} L(\mathcal{A}_{n,d})$ is given by

$$(20) \qquad\qquad B_0/\ldots/B_\ell \mapsto V(B_0) \oplus \ldots \oplus V(B_\ell),$$

where $V(B_0) := \mathbf{0}$, and for any nonzero block $B = \{b_1^{w_1}, b_2^{w_2}, \ldots, b_k^{w_k}\}$ we set

$$V(B) := \{\mathbf{z} \in \mathbb{C}^n \;:\; \zeta^{w_1} z_{b_1} = \zeta^{w_2} z_{b_2} = \ldots = \zeta^{w_k} z_{b_k} \ \text{and} \ z_i = 0 \ \text{for} \ i \notin B\}.$$
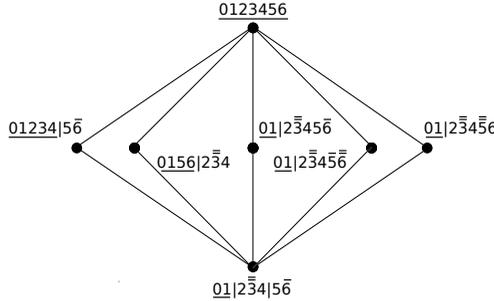
FIGURE 7. $[x, \hat{1}]$ in $\Pi_{6,3}$ for $x = 01|2^0 3^2 4^0|5^0 6^1 = \underline{01}|2\bar{\bar{3}}4|5\bar{6}$ corresponding to $\{\mathbf{z} \in \mathbb{C}^n \;:\; z_1 = 0,\ z_2 = \zeta^2 z_3 = z_4,\ z_5 = \zeta z_6\}$.

Definition 9.2. Let $d > 1$. Call a partition π of $\Pi_{n,d}$ *balanced* if for each block B of π, all weights $w \in \mathbb{Z}/d\mathbb{Z}$ appear in B with equal multiplicity. The $\mathfrak{S}_n$-subposet of $\Pi_{n,d}$ of all balanced partitions is denoted by Π_n^d.
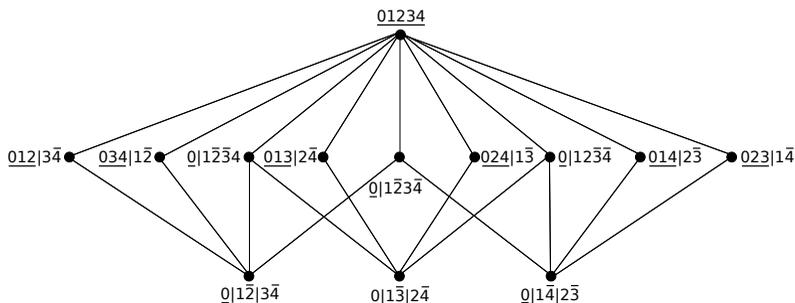
FIGURE 8. Π_4^2 with each weight replaced by an equal number of bars.

Recall that an *upper order ideal* of a poset P is a subposet I with the property that $y \in I$ whenever $x \leq y$ for some $x \in I$. The minimal elements of I are called *generators*. With this terminology, the next observation follows directly from the definitions of $\Pi_{n,d}$ and Π_n^d.

Proposition 9.3. Π_n^d *is the upper order ideal of* $\Pi_{n,d}$ *that is generated by the* $\mathfrak{S}_n$-*orbit of the element* $\pi = B_0/B_1/\ldots/B_{a(d)}$, *where* $a(d) = \left\lfloor \frac{n}{d} \right\rfloor$ *and*

$$B_0 = \{0\} \cup \{\, a(d) \cdot d + 1\, ,\; a(d) \cdot d + 2\, ,\; \ldots\, ,\; n\, \}$$

while

$$B_1 = \{1^0, 2^1, \ldots, d^{d-1}\}$$
$$B_2 = \{(d+1)^0, (d+2)^1, \ldots, (2d)^{d-1}\}$$
$$\vdots$$
$$B_{a(d)} = \{((a(d)-1)\cdot d + 1)^0, \ldots, (a(d)\cdot d)^{d-1}\}.$$

Remark 9.4. Note that $r(\Pi_n^d) = a(d)$ and $\ell(x) = r([x, \hat{1}])$ for $x \in \Pi_n^d$. Also observe that Π_n^d has a bottom element $\hat{0}$ if and only if $n = d = 2$ or $d > n$. In the latter case, the top and bottom elements coincide, i.e., Π_n^d is the trivial poset consisting of the element $\underline{01\cdots n}$.

Theorem 9.5. *Let* $d > 1$. *Then* $E(\mathfrak{S}_n, \zeta) \cong \Pi_n^d$ *as* $\mathfrak{S}_n$-*posets.*

Proof. Because Π_n^d and $E(\mathfrak{S}_n, \zeta)$ are upper order ideals of $\Pi_{n,d}$ and $L(\mathcal{A}_{n,d})$, respectively, it suffices to show that the minimal elements of Π_n^d correspond to those of $E(\mathfrak{S}_n, \zeta)$ under the isomorphism $\Pi_{n,d} \to L(\mathcal{A}_{n,d})$ given in (20). In fact, since the isomorphism is equivariant, by transitivity it suffices to see that the distinguished minimal element of Π_n^d in Proposition 9.3 corresponds to some maximal ζ-eigenspace of $\mathfrak{S}_n$. But this is clear, for the distinguished element corresponds to the ζ-eigenspace of

$$(1, 2, \ldots, d)\,(d+1, d+2, \ldots, 2d)\,\cdots\,(\,(a(d)-1)d+1\, ,\; (a(d)-1)d+2\, ,\; \ldots\, ,\; a(d)d\,),$$

which is maximal under inclusion by considering dimension. $\qquad\square$

A general feature of Dowling lattices is that their upper intervals $[x, \hat{1}]$ are again Dowling lattices; see [12, Thm. 2]. For $x \in \Pi_{n,d}$ one can see this by writing $x = B_0/B_1/\ldots/B_\ell$ and then constructing a poset isomorphism $[x, \hat{1}] \to \Pi_{\ell(x), d}$ by

mapping any block of the form $B_{i_1}^{w_1} \cup \cdots \cup B_{i_j}^{w_j}$ with $w_i \in \mathbb{Z}/d\mathbb{Z}$ to $\{i_1^{w_1}, \ldots, i_j^{w_j}\}$ and mapping any zero block of the form $B_0 \cup B_{i_1}^{\mathrm{erase}} \cup \cdots \cup B_{i_j}^{\mathrm{erase}}$ to zero block $\{0, i_1, \ldots, i_j\}$.

Lemma 9.6. $[x, \hat{1}] \cong \Pi_{\ell(x), d}$ *for each* $x \in \Pi_n^d$.

9.2. **The topology of balanced partitions.** Recall that $E(\mathfrak{S}_n, \zeta) \setminus \{\hat{1}\}$ being shellable implies that its order complex is homotopy equivalent to a bouquet of spheres. The aim of this subsection is to calculate the exact number of such spheres. Central to our analysis is the following consequence of the Hopf trace formula.

Theorem 9.7 (Sundaram [40]). *For P a Cohen–Macaulay G-poset with top element $\hat{1}$, one has an isomorphism of virtual G-modules*

$$\widetilde{H}_{r(P)-1}(P \setminus \{\hat{1}\}) \cong \bigoplus_{i=0}^{r(P)} (-1)^{r(P)+i} \bigoplus_{x^G \in P/G} \mathrm{Ind}_{G_x}^G \widetilde{H}_{i-2}(x, \hat{1}).$$

We shall also need the following consequence of Dowling's well-known description of the Möbius function for any Dowling lattice [12] and Lemma 9.6.

Lemma 9.8. *Let* $x \in \Pi_n^d$. *Then* $\dim \widetilde{H}_{\ell(x)-2}(x, \hat{1}) = \prod_{i=0}^{\ell(x)-1}(1 + id)$.

Regard an *integer partition* λ of $n \geq 0$ as a multiset of positive integers $\lambda_1, \ldots, \lambda_\ell$ with $\sum \lambda_i = n$, and write $|\lambda| = n$ or $\lambda \vdash n$. The integer $\ell(\lambda) := \ell$ is the *length* of λ. We shall also regard the partition λ as a sequence $\lambda = (\lambda_1^{m_1}, \ldots, \lambda_p^{m_p})$ of distinct positive integers $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_p$ with positive weights m_i satisfying $\sum_{i=1}^p m_i \lambda_i = n$. The integer $p(\lambda) := p$ is the number of distinct parts of λ. In what follows, the convention being used will be made clear by the presence or absence of weights m_i.

A *pointed partition* $\Lambda = (\lambda_0, \lambda)$ of $n \geq 0$ is a nonnegative integer $0 \leq \lambda_0 \leq n$ together with a partition λ of $n - \lambda_0$. Such a partition naturally arises in Dowling lattices as the *type* of a partition $x = B_0/B_1/\ldots/B_\ell$, defined to be

$$\mathrm{type}(x) \stackrel{\mathrm{def}}{=} (|B_0 \setminus \{0\}|, \{|B_1|, \ldots, |B_\ell|\}).$$

Corollary 9.9. $\Delta(\Pi_n^d \setminus \{\hat{1}\})$ *is homotopy equivalent to a bouquet of*

$$(21) \qquad \sum_{0 \leq |\lambda| \leq a(d)} \frac{(-1)^{a(d)+\ell(\lambda)}}{d^{\ell(\lambda)}(n - d|\lambda|)!} \frac{n!}{\prod_{j=1}^p \lambda_j!^{m_j d} m_j!} \prod_{i=0}^{\ell(\lambda)-1}(1 + id)$$

many $(a(d) - 1)$*-spheres.*

Proof. Start by noting that two partitions $x, y \in \Pi_n^d$ are in the same $\mathfrak{S}_n$-orbit if and only if $\mathrm{type}(x) = \mathrm{type}(y)$, and observe that for $\mathrm{type}(x) = (\lambda_0, \lambda)$ one has

$$(22) \qquad |\mathrm{Stab}_{\mathfrak{S}_n}(x)| = d^{\ell(\lambda)} \lambda_0! \prod_{i=1}^p m_i!(\lambda_i/d)!^{m_i d}.$$

Taking dimensions in Theorem 9.7 and employing (22) and Lemma 9.8 gives the result. $\qquad\square$

We list some initial values of (21) in Table 5.

$n\backslash d$	2	3	4	5	6	7	8	9
2	0							
3	2	1						
4	1	7	5					
5	21	19	29	23				
6	24	91	89	143	119			
7	510	841	209	503	839	719		
8	918	3529	5251	1343	3359	5759	5039	
9	22246	32367	50275	3023	10079	25919	45359	40319

TABLE 5. $\dim \widetilde{H}_{a(d)-1}(\Pi_n^d \setminus \{\hat{1}\})$ for $n \leq 9$ and $a(d) \neq 0$.

Remark 9.10. One may identify the stabilizer $\mathrm{Stab}_{\mathfrak{S}_n}(x)$ of $x = B_0/B_1/\ldots/B_\ell$ as a (suitably defined) product of nested wreath products

$$\mathfrak{S}_{\lambda_0} \times \mathfrak{S}_{m_1}[C_d][\mathfrak{S}_{\lambda_1/d}] \times \ldots \times \mathfrak{S}_{m_p}[C_d][\mathfrak{S}_{\lambda_p/d}],$$

from which one can show that

$$\widetilde{H}_{\ell-2}(x, \hat{1}) \cong_{\mathrm{Stab}_{\mathfrak{S}_n}(x)} \mathrm{Res}^{\mathfrak{S}_{\lambda_0} \times \mathfrak{S}_\ell[C_d][\mathfrak{S}_n]}_{\mathrm{Stab}_{\mathfrak{S}_n}(x)} \mathbb{1} \otimes \widetilde{H}_{\ell-2}(\overline{\Pi}_{\ell,d})[\mathbb{1}]$$

for a suitable embedding $\mathrm{Stab}_{\mathfrak{S}_n}(x) \subseteq \mathfrak{S}_{\lambda_0} \times \mathfrak{S}_\ell[C_d][\mathfrak{S}_n]$. Here, $\widetilde{H}_{\ell-2}(\overline{\Pi}_{\ell,d})[\mathbb{1}]$ denotes the $\mathfrak{S}_\ell[C_d][\mathfrak{S}_n]$-module $\mathbb{1}^{\otimes \ell d} \otimes \widetilde{H}_{\ell-2}(\overline{\Pi}_{\ell,d})$ given by

$$(\sigma_1, \ldots, \sigma_{\ell d} ; \sigma)(v_1 \otimes \cdots \otimes v_n \otimes w) = v_1 \otimes \cdots \otimes v_n \otimes \sigma w$$

for $\sigma_i \in \mathfrak{S}_n$ and $\sigma \in \mathfrak{S}_\ell[C_d]$, and $\Pi_{\ell,d}$ is viewed as a $\mathfrak{S}_\ell[C_d]$-poset. Employing Theorem 9.7, one then obtains a (virtual) expression for $\widetilde{H}_{a(d)-1}(\Pi_n^d \setminus \{\hat{1}\})$ as an $\mathfrak{S}_n$-module in terms of the $\mathfrak{S}_\ell[C_d]$-modules $\widetilde{H}_{\ell-2}(\overline{\Pi}_{\ell,d})$ studied by Hanlon [18] and Gottlieb–Wachs [16].

Problem 9.11. Describe the $\mathfrak{S}_n$-irreducible decomposition of $\widetilde{H}_{a(d)-1}(\Pi_n^d \setminus \{\hat{1}\})$ explicitly in general.

Tables 6–9 give the decomposition of $\widetilde{H}_{a(d)-1}(\Pi_n^d \setminus \{\hat{1}\})$ into irreducible submodules for $n = 4, 5, 6, 7$. Each entry lists the multiplicity with which the Specht module S^λ occurs in $\widetilde{H}_{a(d)-1}(\Pi_{|\lambda|}^d \setminus \{\hat{1}\})$.

$d\backslash\lambda$	⊞⊟	⊞	⊟	⊟
2				1
3	1		1	1
4		1	1	

TABLE 6. Decomposition of the $\mathfrak{S}_4$-module $\widetilde{H}_{a(d)-1}(\Pi_4^d \setminus \{\hat{1}\})$.

(Column headers marked $\lambda^{(i)}$ are Young diagrams.)

$d \backslash \lambda$	$\lambda^{(1)}$	$\lambda^{(2)}$	$\lambda^{(3)}$	$\lambda^{(4)}$	$\lambda^{(5)}$	$\lambda^{(6)}$
2		1	1	1	1	1
3	1	1	1		1	
4	1	1	1	2	1	
5		1	2	1		1

TABLE 7. Decomposition of the $\mathfrak{S}_5$-module $\widetilde{H}_{a(d)-1}(\Pi_5^d \setminus \{\hat{1}\})$.

$d \backslash \lambda$	$\lambda^{(1)}$	$\lambda^{(2)}$	$\lambda^{(3)}$	$\lambda^{(4)}$	$\lambda^{(5)}$	$\lambda^{(6)}$	$\lambda^{(7)}$	$\lambda^{(8)}$	$\lambda^{(9)}$	$\lambda^{(10)}$
2						1		1	1	
3			1		2	2	1	2	1	1
4	1	2	1		2	1	1	1		
5	1	1	2	1	4	2	1	1	1	1
6		2	2	1	2	2	2	1	1	

TABLE 8. Decomposition of the $\mathfrak{S}_6$-module $\widetilde{H}_{a(d)-1}(\Pi_6^d \setminus \{\hat{1}\})$.

$d \backslash \lambda$	$\lambda^{(1)}$	$\lambda^{(2)}$	$\lambda^{(3)}$	$\lambda^{(4)}$	$\lambda^{(5)}$	$\lambda^{(6)}$	$\lambda^{(7)}$	$\lambda^{(8)}$	$\lambda^{(9)}$	$\lambda^{(10)}$	$\lambda^{(11)}$	$\lambda^{(12)}$	$\lambda^{(13)}$	$\lambda^{(14)}$
2				1	2	2	2	2	5	3	2	3	2	
3		1	2	1	5	4	3	3	7	4	3	3	2	1
4	1	2	1	1	2	1		1	1					
5	1	2	2	1	4	2	3	2	3	1	1		1	
6	1	2	2	3	6	4	3	4	5	3	3	2	1	
7		2	3	2	5	2	3	3	5	3	2	2		1

TABLE 9. Decomposition of the $\mathfrak{S}_7$-module $\widetilde{H}_{a(d)-1}(\Pi_7^d \setminus \{\hat{1}\})$.

9.3. **The d-divisible partition poset.** Throughout this section we shall assume that $d > 1$. For $n+1$ divisible by d, recall that a set partition of $\{1, 2, \ldots, n+1\}$ is called *d-divisible* if all blocks have size divisible by d, and let P_{n+1}^d denote the $\mathfrak{S}_{n+1}$-poset of all such partitions ordered by refinement. The *d-divisible partition lattice* $P_{n+1}^d \cup \{\hat{0}\}$ has been extensively studied, starting with Sylvester [42] computing the Möbius function for $d = 2$. Stanley [37] generalized Sylvester's result to arbitrary d and conjectured that the restriction of the top (reduced) homology module of $P_{n+1}^d \setminus \{\hat{1}\}$ to $\mathfrak{S}_n$ is the Specht module of skew ribbon shape $(d - 1, d, d, \ldots, d)$; see Figure 9. (One instead obtains the regular representation of $\mathfrak{S}_n$ if $d = 1$; see [38].) Calderbank–Hanlon–Robinson [9] established Stanley's conjecture via character calculations, and Wachs [44] later exhibited an explicit basis for the top homology group via an EL-shelling. Ehrenborg and Jung [13] extended this phenomenon to more general posets of *pointed partitions* $\Pi_{\vec{c}}^{\bullet}$, showing that each $\Pi_{\vec{c}}^{\bullet} \setminus \{\hat{1}\}$ is either contractible or has top homology the $\mathfrak{S}_n$-Specht module of skew ribbon shape associated with the composition $\vec{c} = (c_1, c_2, \ldots, c_k)$.
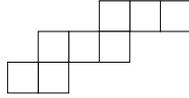
FIGURE 9. Skew ribbon shape $(2, 3, 3)$.

In this section we present an intermediate family of posets $\Pi_n^{(d)}$, between the family of d-divisible partition posets P_{n+1}^d and the family of pointed partition posets $\Pi_{\vec{c}}^\bullet$, that arises naturally in the context of eigenspace arrangements for $\mathfrak{S}_n$. Furthermore, we conjecture that the top homology of $\Pi_n^{(d)} \setminus \{\hat{1}\}$ is isomorphic to a submodule of the top homology of $\Pi_n^d \setminus \{\hat{1}\}$.

For a pointed partition $\Lambda = (\lambda_0, \lambda)$ of n, let $\Pi_\Lambda \subseteq \Pi_{n,1}$ denote the upper order ideal generated by the elements of type Λ and regard it as an $\mathfrak{S}_n$-poset through the usual action of $\mathfrak{S}_n$ on $[n]$. For the particular choice

$$\Lambda = (n - d \cdot a(d) \ , \ \{d, d, \ldots, d\}),$$

write $\Pi_n^{(d)}$ for Π_Λ and call it the *pointed d-divisible partition poset*. It is $\mathfrak{S}_n$-isomorphic to Ehrenborg and Jung's poset $\Pi_{(d,d,\ldots,d,n-d\cdot a(d))}^\bullet$ via the map that removes all weights and then replaces each zero block B_0 with the *distinguished block* $Z := B_0 \setminus \{0\}$; see [13] for details. For $n + 1$ divisible by d, the map

(23) $0 \mapsto n + 1$ and $B \mapsto B^{\mathrm{erase}}$ for each nonzero block B

extends to an $\mathfrak{S}_n$-isomorphism $\Pi_n^{(d)} \to P_{n+1}^d$ when P_{n+1}^d is considered as an $\mathfrak{S}_n$-poset by restricting its natural $\mathfrak{S}_{n+1}$-action to $\mathfrak{S}_{\{1,\ldots,n\}}$. As a consequence of this discussion and the main results of [13] we have the following.

Theorem 9.12 (Calderbank–Hanlon–Robinson, Ehrenborg–Jung, Wachs).
$\widetilde{H}_{a(d)-1}(\Pi_n^{(d)} \setminus \{\hat{1}\})$ *is isomorphic to the $\mathfrak{S}_n$-Specht module of skew ribbon shape associated with* $(n - d \cdot a(d), d, d, \ldots, d)$ *when $d \nmid n$, and 0 otherwise.*

Remark 9.13. We have stated Theorem 9.12 in a way that requires the observation that the skew shapes associated to $(n - d \cdot a(d), d, d, \ldots, d)$ and its reverse $(d, d, \ldots, d, n - d \cdot a(d))$ are related by 180° rotation, and hence their Specht modules are $\mathfrak{S}_n$-isomorphic; see [36, Exercise 7.56].
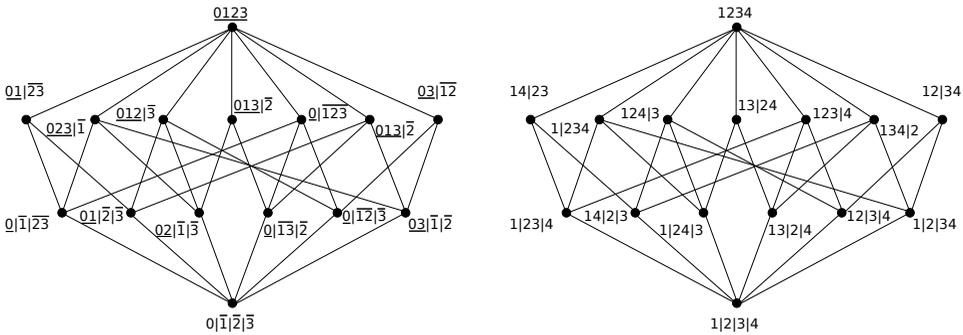
FIGURE 10. The posets $\Pi_{n,1}$ and Π_{n+1} when $n = 3$.

Note that the $\mathfrak{S}_n$-ribbon representation of Theorem 9.12 arises from a subarrangement of the reflection arrangement of $\mathfrak{S}_{n+1}$:

$$\Pi_n^{(d)} \subseteq \Pi_{n,1} \cong_{\mathfrak{S}_n} \Pi_{n+1} \cong_{\mathfrak{S}_n} E(\mathfrak{S}_{n+1}, 1) \quad \text{with} \quad \mathfrak{S}_n = \mathfrak{S}_{\{1,\ldots,n\}},$$

where the isomorphism $\Pi_{n,1} \cong_{\mathfrak{S}_n} \Pi_{n+1}$ is obtained by extending (23); see Fig. 10. We conjecture that these $\mathfrak{S}_n$-ribbon representations also appear in $E(\mathfrak{S}_n, \zeta_d)$, without the (a priori) mysterious shift of index. To make the assertion more precise, we introduce the new operation $B \mapsto B^{\mathrm{zero}}$ of *zeroing*: given a block B, define B^{zero} to be the block obtained from B by replacing each weight w with 0. An immediate observation is the following; see Figure 11.

Proposition 9.14. *The map* $Z : \Pi_n^d \to \Pi_n^{(d)} \subseteq \Pi_{n,1}$ *given by zeroing each block*

$$Z : B_0/B_1/\ldots/B_\ell \mapsto B_0/B_1^{\mathrm{zero}}/\ldots/B_\ell^{\mathrm{zero}}$$

is a rank-preserving surjective $\mathfrak{S}_n$-poset map.
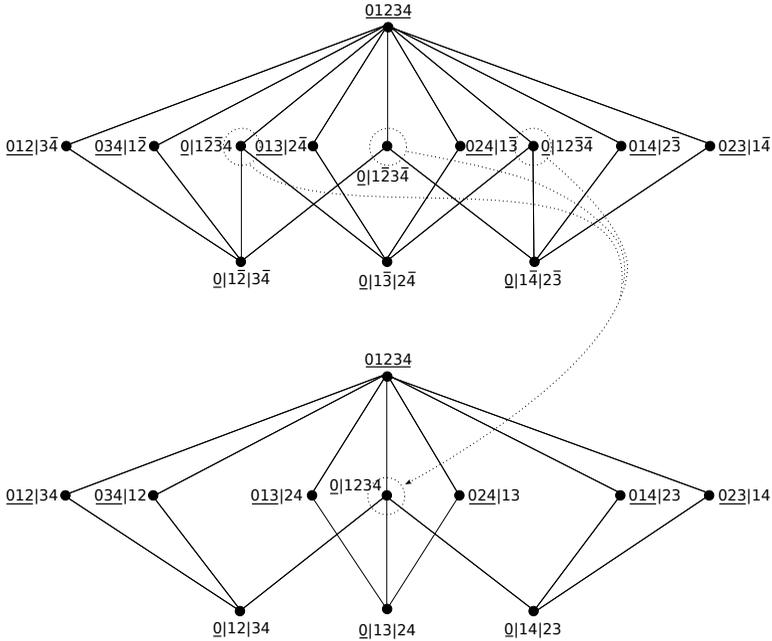
FIGURE 11. The zeroing map $\Pi_4^2 \to \Pi_4^{(2)}$.

Conjecture 9.15. *Let* $Z_* : \widetilde{H}_{a(d)-1}(\Pi_n^d \setminus \{\hat{1}\}) \to \widetilde{H}_{a(d)-1}(\Pi_n^{(d)} \setminus \{\hat{1}\})$ *be the homomorphism induced by the poset map of Proposition 9.14. Then Z_* is surjective.*

Note that by Theorem 9.12 the conjecture is trivially true when $d \mid n$. The conjecture also clearly holds when $r(\Pi_n^d) = 1$, i.e., when $a(d) = 1$. In fact, the entire top homology module is straightforward in this case:

Proposition 9.16. *Suppose $a(d) = 1$. Let C_d be the cyclic group generated by the d-cycle $(1, 2, \ldots, d)$ and let $\mathfrak{S}_{n-d} = \mathfrak{S}_{\{d+1, d+2, \ldots, n\}}$. Then*

$$\widetilde{H}_{a(d)-1}(\Pi_n^d \setminus \{\hat{1}\}) \cong_{\mathfrak{S}_n} \left(\mathrm{Ind}_{C_d \times \mathfrak{S}_{n-d}}^{\mathfrak{S}_n} \mathbb{1} \right) - \mathbb{1}.$$

Proof. $\mathfrak{S}_n$ acts transitively on the elements of $\Pi_n^d \setminus \{\hat{1}\}$ in this case, and $C_d \times \mathfrak{S}_{n-d}$ is the stabilizer of the distinguished element of Proposition 9.3. Hence the result. $\square$

10. Proof of Theorem 1.3

Here we characterize when eigenspace arrangements are actually hyperplane arrangements (Lemma 10.3 below). Then as a consequence of the characterization, Theorem 1.3 follows from results of Bessis and Terao.

The motivating case $d = 1$ of Theorem 1.3 has a long history, starting with Fadell and Neuwirth [14] showing that $\mathcal{M}(\mathfrak{S}_n, 1)$ is $K(\pi, 1)$. Deligne [11] later proved Brieskorn's conjecture that $\mathcal{M}(W, 1)$ is $K(\pi, 1)$ for all (complexified) real reflection groups, then Orlik–Solomon [30] showed that $\mathcal{M}(W, 1)$ is $K(\pi, 1)$ for every *Shephard group* W (see [31, §6.6]) by showing that $\mathcal{M}(W, 1)/W$ is the same as $\mathcal{M}(W', 1)/W'$ for some associated real reflection group W' and invoking Deligne's result, while Nakamura [28] established the result for the family $G(m, p, n)$, leaving only six exceptional cases. Bessis [2] resolved the question in the remaining cases, while also recovering many of the previous results with his approach. In particular, he obtained a new proof of Deligne's result for real reflection groups.

Theorem 10.1 (Bessis)**.** *For any reflection group W, the reflection complement $\mathcal{M}(W, 1)$ is $K(\pi, 1)$, i.e., its universal cover is contractible.*

The most one could hope for in general is that $\mathcal{M}(W, d)$ is $K(\pi, 1)$ whenever $a(d)$ is n or $n - 1$, since any complex arrangement $\mathcal{A}$ whose complement is $K(\pi, 1)$ must contain a hyperplane; see [7, Cor. 3.2] for a much stronger result. Since Proposition 3.7(i) tells us that $\mathcal{A}(W, d)$ is in fact a hyperplane arrangement when $a(d)$ is n or $n - 1$, one may also hope it is *free* in the sense of Terao [43], who established freeness for reflection arrangements $\mathcal{A}(W, 1)$.

Theorem 10.2 (Terao)**.** *For any reflection group W, the reflection arrangement $\mathcal{A}(W, 1)$ is free.*

Recall that a collection of hyperplanes $\mathcal{A}$ in a vector space V is a *free arrangement* if the S-module of derivations $\delta : S \to S$ that satisfy $\delta(\alpha_H) \in \alpha_H S$ for every hyperplane $H = \ker(\alpha_H)$ in $\mathcal{A}$ is free, where $S = S(V^*)$ as in §3. In particular, the *null arrangement* $\mathcal{N}_V$ of V, which contains no subspaces at all, is a free arrangement. Less obvious is that any finite collection of lines through the origin in $\mathbb{C}^2$ is free, which follows from *Saito's criterion.* (It is also straightforward [31, Examples 4.20 and 5.4] to see that they are $K(\pi, 1)$, by which we mean that their complements have the property.) Lastly, recall that the product

$$\mathcal{A}_1 \times \mathcal{A}_2 := \{H_1 \oplus V_2 \ : \ H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 \ : \ H_2 \in \mathcal{A}_2\}$$

of two arrangements is free if and only if both $\mathcal{A}_1$ and $\mathcal{A}_2$ are free; see [31].

We first characterize when $\mathcal{A}(W, d)$ is a hyperplane arrangement.

Lemma 10.3. *Let W be a complex reflection group and let $d > 0$.*
  (i) If $a(d) = n$, then $\mathcal{A}(W, d) = \mathcal{A}(W, 1)$ by Corollary 3.8.
  (ii) If $a(d) = n - 1$ and W is irreducible, then one of the following holds:
     a. W has rank 2, so that $\mathcal{A}(W, d)$ is a set of lines through the origin in $\mathbb{C}^2$.
     b. $W = G_{25}$, $d = 2$ (or 6), and $\mathcal{A}(W, d) = \mathcal{A}(W', 1)$ for $W' = G(3, 3, 3)$.
     c. $W = W(A_3)$, $d = 2$, and $\mathcal{A}(W, d)$ is defined by $z_1 z_2 z_3$.
     d. $W = G(m, p, n)$, $d \mid m$ and $d \nmid \frac{nm}{p}$, and $\mathcal{A}(W, d)$ is defined by $z_1 z_2 \cdots z_n$.

Proof. (i) is clear. Consulting the classification [34], one finds that the pairs (W, d) listed in (ii) are the only cases such that $a(d) = n - 1$. Case (iia) is clear. For (iic) and (iid), set $f_i := e_i(z_1^m, z_2^m, \ldots, z_n^m)$ for $1 \le i \le n - 1$ and set $f_n := e_n(z_1^{m/p}, z_2^{m/p}, \ldots, z_n^{m/p})$, where here e_i denotes the *i*th *elementary symmetric polynomial* in n variables. These form a set of basic invariants for $G(m, p, n)$, the degrees of which are $m, 2m, \ldots, (n-1)m, nm/p$, and only the last invariant polynomial $f_n = (z_1 z_2 \cdots z_n)^{m/p}$ has degree $d_n = nm/p$ not divisible by d. Applying Proposition 3.5 thus gives (iic) and (iid). Similarly, for G_{25} we may choose [26, Eq. 9] coordinates and basic invariants f_1, f_2, f_3 (of degrees 6,9,12) such that f_2 is the polynomial $(z_1^3 - z_2^3)(z_1^3 - z_3^3)(z_2^3 - z_3^3)$ that (by (5)) defines $\mathcal{A}(W', 1)$, then apply Proposition 3.5. $\qquad\square$

Now Theorem 1.3 follows from the results of Bessis and Terao.

Proof of Theorem 1.3. As noted above, if $\mathcal{A}(W, d)$ is $K(\pi, 1)$ (resp. free), then either $a(d) = n$ or $a(d) = n - 1$. When $a(d) = n$, the converse follows from Theorem 10.1 (resp. Theorem 10.2), so it remains to see that the converse also holds when $a(d) = n - 1$. In this case, decompose W and $V := \mathbb{C}^n$ as in Proposition 3.1 so that $W \cong W_1 \times \cdots \times W_k$. Then $\mathcal{A}(W, d) = \mathcal{A}(W_i, d) \times \mathcal{N}_{V_i^\perp}$ for some index i, and thus $\mathcal{A}(W, d)$ is $K(\pi, 1)$ (resp. free) if and only if $\mathcal{A}(W_i, d)$ is $K(\pi, 1)$ (resp. free). Now employ Lemma 10.3(ii). $\qquad\square$

Remark 10.4. V. Reiner pointed out that one may alternatively consider the question of real $K(\pi, 1)$ complements. That is, for which (uncomplexified) real reflection groups $W \subset \mathrm{GL}(\mathbb{R}^n)$ is the *real* complement

$$\mathcal{M}_{\mathbb{R}}(W, 2) \overset{\mathbf{def}}{=} \mathbb{R}^n \setminus \bigcup_{g \in W} V(g, -1)$$

a $K(\pi, 1)$ space? Because such a complement is $K(\pi, 1)$ only if the arrangement has codimension 2, one reduces the question to Weyl groups $W(A_4)$, $W(A_5)$, and $W(E_6)$ via the classification. Though we have not explored these cases, we note that a similar question was answered in [10], where Davis, Januszkiewicz, and Scott established a conjecture of Khovanov [20] asserting that the real complement of any W-invariant codimension-2 subarrangement of $\mathscr{L}_W$ is $K(\pi, 1)$ when W is a real reflection group. Björner first suggested looking at real $K(\pi, 1)$ complements, and Khovanov was motivated by (and answered positively) Björner's question [3, §13.7] which asked whether the complement of the 3-*equal arrangement*

$$\mathcal{A} = \{\mathbf{x} \in \mathbb{R}^n \ : \ x_i = x_j = x_k \quad \text{for some triple} \quad i < j < k\}$$

is a $K(\pi, 1)$ space.

W	$a(d)$	d	$W(d)$	Rank sizes of $E(W,\zeta_d)$					Cell counts of $\Delta(\overline{E}(W,\zeta_d))$				
				r_0	r_1	r_2	r_3	r_4	$|C_{-1}|$	$|C_0|$	$|C_1|$	$|C_2|$	$|C_3|$
G_{25}	2	2,6	G_5	9	48	1			1	57	72		
G_{28}	2	3,6	G_5	16	64	1			1	80	128		
	2	4	G_8	12	36	1			1	48	72		
G_{30}	2	3,6	G_{20}	40	400	1			1	440	800		
	2	4	G_{22}	60	900	1			1	960	1800		
	2	5,10	G_{16}	24	144	1			1	168	288		
G_{31}	2	3,6,12	G_{10}	160	880	1			1	1040	2240		
	2	8	G_9	240	1800	1			1	2040	4320		
G_{32}	2	4,12	G_{10}	540	1980	1			1	2520	7560		
G_{33}	2*	4	G_6	540	1860	1			1	2400	5400		
	3	3,6	G_{26}	40	600	925	1		1	1565	7920	6720	
G_{34}	2*	4,12	G_{10}	34020	47250	1			1	13230	68040		
G_{35}	2	4	G_8	540	540	1			1	1080	3240		
	2	6	G_5	720	1680	1			1	2400	5760		
(type E_6)	3	3	G_{25}	80	480	330	1		1	890	5040	4800	
	4	2	G_{28}	45	810	2730	1611	1	1	5196	58590	130680	77760
G_{36} (type E_7)	2*	4	G_8	11340	1890	1			1	13230	68040		
	3	3,6	G_{26}	2240	18480	13272	1		1	33992	322560	376320	
G_{37} (type E_8)	2	5,10	G_{16}	1161216	?	1			1	?	$11 \times r_0$		
	2	8	G_9	3628800	?	1			1	?	$18 \times r_0$		
	2	12	G_{10}	2419200	?	1			1	?	$14 \times r_0$		
	4	3,6	G_{32}	4480	89600	319200	103040	1	1	$\sum_0^3 r_i$	?	?	$6720 \times r_0$
	4	4	G_{31}	15120	453600	2205000	1519560	1	1	$\sum_0^3 r_i$	?	?	$22680 \times r_0$

TABLE 10. An expanded version of Table 2.

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