VANISHING RESULTS FOR CHARACTER TABLES

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Interest in zeros of characters goes back to the beginning of character theory with Burnside's result that each nonlinear irreducible character χ of a finite group G has zeros. But how many zeros? In particular, we ask the following [3].

Question 1. What is the chance that a character value $\chi(g)$ equals 0?

The two most natural ways of choosing a character value $\chi(g)$ are as follows.

1) Choose $\chi \in \mathrm{Irr}(G)$ and $g \in G$ uniformly at random, and then evaluate $\chi(g)$. The chance that $\chi(g)$ equals zero will be denoted by $\mathrm{Prob}(\chi(g) = 0)$, so

$$\mathrm{Prob}(\chi(g) = 0) = \frac{|\{(\chi, g) \in \mathrm{Irr}(G) \times G : \chi(g) = 0\}|}{|\mathrm{Irr}(G) \times G|}.$$

2) Choose $\chi \in \mathrm{Irr}(G)$ and a class $K = g^G \in \mathrm{Cl}(G)$ uniformly at random, and then evaluate $\chi(K) := \chi(g)$. In other words, choose an entry $\chi(K)$ uniformly at random from the character table of G. The chance that $\chi(K)$ equals zero will be denoted by $\mathrm{Prob}(\chi(K) = 0)$, so

$$\mathrm{Prob}(\chi(K) = 0) = \frac{|\{(\chi, K) \in \mathrm{Irr}(G) \times \mathrm{Cl}(G) : \chi(K) = 0\}|}{|\mathrm{Irr}(G) \times \mathrm{Cl}(G)|}.$$

Example 2. If $G = S_4$, then

$$\mathrm{Prob}(\chi(g) = 0) = \frac{28}{120} \approx 0.194 \quad \text{and} \quad \mathrm{Prob}(\chi(K) = 0) = \frac{4}{25} = 0.16.$$

§1. It turns out that many characters have many zeros. The first result in this direction is for symmetric groups [3, Theorem 1].

Theorem 3. *If $\chi \in \mathrm{Irr}(S_n)$ and $g \in S_n$ are chosen uniformly at random, then $\chi(g) = 0$ with probability $\to 1$ as $n \to \infty$.*

One of the two proofs given in [3] proceeds by showing that a vanishingly small fraction of $\mathrm{Cl}(S_n)$ covers almost all of S_n. The key ingredient here is the following inequality [3, Proposition 3].

Lemma 4. *For any finite group G and any collection $\mathcal{K} \subseteq \mathrm{Cl}(G)$,*

$$\mathrm{Prob}(\chi(g) = 0) \geq \frac{|\{g \in G : g^G \in \mathcal{K}\}|}{|G|} - \frac{|\mathcal{K}|}{|\mathrm{Cl}(G)|}.$$

Different groups require different tools. The following bound in terms of character degrees and class sizes was established in joint work of P. X. Gallagher, M. J. Larsen, and the speaker [1].

Lemma 5. *For each finite group G and each $\epsilon > 0$,*

$$\mathrm{Prob}(\chi(g) \neq 0) \leq \frac{|\{(\chi, g) \in \mathrm{Irr}(G) \times G : \gcd(\chi(1), |g^G|) \geq \epsilon\chi(1)\}|}{|\mathrm{Irr}(G) \times G|} + \epsilon^2.$$

Lemma 5 was used in [1] to establish the $\mathrm{GL}(n,q)$ analogue of Theorem 3.

Theorem 6. *For $G = \mathrm{GL}(n,q)$, the proportion $P_{n,q}$ of pairs $(\chi, g) \in \mathrm{Irr}(G) \times G$ with $\chi(g) = 0$ satisfies*

$$\inf_q P_{n,q} \to 1 \text{ as } n \to \infty.$$

So for any sequence of prime powers $q_1, q_2, \ldots$, we have $P_{n,q_n} \to 1$ as $n \to \infty$.

We also have the following result about the sparsity of the character tables of finite simple groups of Lie type due to M. Larsen and the speaker [2, Theorem 1.1].

Theorem 7. *If G_n is any sequence of finite simple groups of Lie type with rank tending to ∞, then almost every entry in the character table of G_n is zero as $n \to \infty$. In other words, the fraction of the character table of G_n that is covered by zeros tends to 1 as $n \to \infty$.*

For S_n, however, we do not know the limiting behavior of $\mathrm{Prob}(\chi(K) = 0)$, which we shall denote by $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$ with the understanding that λ and μ are chosen uniformly at random from the partitions of n and $\chi_\lambda(\mu)$ is shorthand for the value of χ_λ at any permutation of cycle type μ. See [3] and [4, Table 3]. The following question [3, Question 2] is wide open.

Question 8. What can be said about the limiting behavior of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$?

§2. Instead of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$, what happens if we work mod 2? One might guess that $\mathrm{Prob}(\chi_\lambda(\mu) \equiv 0 \mod 2) \to 1/2$ as $n \to \infty$. And for $n = 4, 5, \ldots, 10$, the proportions are approximately 0.24, 0.33, 0.36, 0.40, 0.55, 0.56, 0.55. But the values keep growing. At $n = 76$, for example, roughly 87% of the more than 86 trillion entries in the character table of S_n are even. See [4] for more data. The speaker also carried out computations for other primes and prime powers and conjectured that, for any positive integer m, $\mathrm{Prob}(\chi_\lambda(\mu) \equiv 0 \mod m) \to 1$. After several partial results by various authors, this conjecture is now known to be true.

§3. Let $P(G)$ denote a fixed choice of either $\mathrm{Prob}(\chi(g) = 0)$ or $\mathrm{Prob}(\chi(K) = 0)$. So far, we have discussed starting with a sequence of groups $G_1, G_2, \ldots$ and then studying the corresponding fractions $P(G_1), P(G_2), \ldots \in [0, 1]$.

3.1. $P(G)$ can also be studied as a random variable itself, with G chosen from some distribution. For example, we can choose a random Young subgroup S_λ of S_n. The natural question is then: What is the expected value of $P(S_\lambda)$ when λ is chosen uniformly at random from the partitions of n? The answer is the following [5, Theorem 2].

Theorem 9. *The expected value of $P(S_\lambda)$ tends to 1 as $n \to \infty$.*

3.2. We can also ask what the sequences $P(G_1), P(G_2), \ldots$ can possibly look like. The following answer is Theorem 1 in [5].

Theorem 10. *If $a_1, a_2, \ldots \in [0, 1]$ and $\epsilon_1, \epsilon_2, \ldots \in (0, \infty)$, then for each prime p there exists an ascending chain of p-groups $G_1 < G_2 < \ldots$ such that, for each i,*

$$|P(G_i) - a_i| < \epsilon_i.$$

In particular, the set $\{P(G) : |G| < \infty\}$ is dense in $[0, 1]$.

§4. There are also the classical results of J. G. Thompson and P. X. Gallagher involving both zeros and roots of unity. Thompson proved that if $\chi \in \mathrm{Irr}(G)$, then $\chi(g)$ is either zero or a root of unity for more than a third of the elements $g \in G$. In terms of the function

$$\theta(G) = \min_{\chi \in \mathrm{Irr}(G)} \frac{|\{g \in G : \chi(g) \text{ is zero or a root of unity}\}|}{|G|},$$

Thompson's result says

$$\theta(G) > 1/3.$$

Gallagher proved similarly that if K is a larger than average conjugacy class of a finite group G, then $\chi(K)$ is either zero or a root of unity for more than a third of the characters $\chi \in \mathrm{Irr}(G)$. In terms of the function

$$\theta'(G) = \min_{K} \frac{|\{\chi \in \mathrm{Irr}(G) : \chi(K) \text{ is zero or a root of unity}\}|}{|\mathrm{Irr}(G)|},$$

with the minimum taken over all larger than average conjugacy classes K of G, Gallagher's result says

$$\theta'(G) > 1/3.$$

We ask in [6] if the Thompson and Gallagher lower bounds are the best possible. More specifically, we ask the following [6, Questions 1 and 2].

Question 11. What are the greatest lower bounds $\inf_G \theta(G)$ and $\inf_G \theta'(G)$?

The main conjecture of [6] is the following answer.

Conjecture 12. $\inf \theta(G) = 1/2$ and $\inf \theta'(G) = 1/2$.

The greatest lower bounds can not be greater than 1/2 [6, Eqs. (20) and (21)].

Proposition 13. *For $G_n = Suz(2^{2n+1})$, we have $\theta(G_n) \to \frac{1}{2}^+$ and $\theta'(G_n) \to \frac{1}{2}^+$.*

So Conjecture 12 is equivalent to the following [6, Conjecture 1].

Conjecture 14. $\theta(G) \geq 1/2$ and $\theta'(G) \geq 1/2$ for every finite group G.

As evidence for the conjecture, we have the following [6, Cor. 3 and Thm. 10].

Theorem 15. *Conjecture 14 holds for the following groups.*
- *All finite groups of order $< 2^9$.*
- *All simple groups of order $\leq 10^9$.*
- *All sporadic groups.*
- *A_n, $L_2(q)$, $Suz(2^{2n+1})$, $Ree(3^{2n+1})$.*
- *All finite nilpotent groups.*

In fact for finite nilpotent groups we have the following much stronger results [6, Theorems 1 and 2].

Theorem 16. *Each nonlinear irreducible character of a finite nilpotent group is zero on more than half of the group elements.*

Theorem 17. *More than half of the nonlinear irreducible characters of a finite nilpotent group are zero on any given larger than average class.*

The main new ingredient for these vanishing results is [6, Theorem 8], which is a very strong improvement of a classical result of Siegel [7] about totally positive algebraic integers, but in the context of the totally positive integers $|\chi(g)|^2$ coming from finite nilpotent groups.

References

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